ON THE NOTIONS OF UPPER AND LOWER DENSITY

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Abstract. Let $\mathcal{P}(\mathbf{N})$ be the power set of $\mathbf{N}$. We say that a function $\mu^\star : \mathcal{P}(\mathbf{N}) \to \mathbf{R}$ is an upper density if, for all $X, Y \subseteq \mathbf{N}$ and $h, k \in \mathbf{N}^+$, the following hold: (F1) $\mu^\star(\mathbf{N}) = 1$; (F2) $\mu^\star(X) \le \mu^\star(Y)$ if $X \subseteq Y$; (F3) $\mu^\star(X \cup Y) \le \mu^\star(X) + \mu^\star(Y)$; (F4) $\mu^\star(k \cdot X) = \frac{1}{k}\mu^\star(X)$; (F5) $\mu^\star(X + h) = \mu^\star(X)$, where, in particular, $k \cdot X := \{kx : x \in X\}$.

We show that the upper asymptotic, upper logarithmic, upper Banach, upper Buck, upper Pólya, and upper analytic densities, together with all the upper α-densities (with α a real parameter ≥ -1), are examples of upper densities in the sense of the definition above.

Moreover, we establish the mutual independence of axioms (F1)-(F5) and investigate various properties of upper densities and some related functions.

These properties, most of which are in fact proved under the assumption that axiom (F2) is replaced by the weaker condition that $\mu^\star(X) \le 1$ for every $X \subseteq \mathbf{N}$, extend and generalize results so far independently derived for some of the classical upper densities we mentioned before, thus introducing a certain amount of unification into the theory.

1. Introduction

Densities have played a fundamental role in the development of (probabilistic and additive) number theory and certain areas of analysis and ergodic theory, as witnessed by the great deal of research on the subject, see, e.g., [26], [29], [19], [46, Part III, § 1.1], [28, Part III], [3], [38, Chapter 4], [20], [22], and references therein. One reason is that densities provide an interesting alternative to measures when it comes to the problem of studying the interrelation between the "structure" of a set of integers X and some kind of information about the "largeness" of X.

This principle is fully embodied in the Erdős-Turán conjecture [44, § 35.4] that any set X of positive integers such that $\sum_{x \in X} \frac{1}{x} = \infty$ contains arbitrarily long finite arithmetic progressions, two celebrated instances of which are Szemerédi's theorem on sets of positive upper asymptotic density [45] and the Green-Tao theorem on the primes [13].

The present paper fits into this context, insofar as we aim at characterizing the upper asymptotic and upper Banach density as two of the uncountably many functions satisfying a suitable set of conditions that we use to give a more conceptual proof of some nontrivial properties of these and other "upper densities" (see § 4 and Example 8 for a few samples).

An analogous point of view was picked up, for instance, by A. R. Freedman and J. J. Sember (motivated by the study of convergence in sequence spaces) in [8], where a lower density (on

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$\mathbf{N}^+$) is abstractly defined as a nonnegative (set) function $\delta_\star : \mathcal{P}(\mathbf{N}^+) \to \mathbf{R}$ such that, for all $X, Y \subseteq \mathbf{N}^+$, the following hold (see § 1 for notation and terminology):

(L1) $\delta_\star(X) + \delta_\star(Y) \le \delta_\star(X \cup Y)$ if $X \cap Y = \varnothing$;
(L2) $\delta_\star(X) + \delta_\star(Y) \le 1 + \delta_\star(X \cap Y)$;
(L3) $\delta_\star(X) = \delta_\star(Y)$ provided that $|X \triangle Y| < \infty$;
(L4) $\delta_\star(\mathbf{N}^+) = 1$.

(In fact, Freedman and Sember assume that the codomain of $\delta_\star$ is the interval $[0, 1]$, which, however, does not make any substantial difference for their work, as it follows from the definition above that $0 \le \delta_\star(X) \le 1$ for every $X \subseteq \mathbf{N}^+$.) Then it is possible to define an upper density, herein referred to as the conjugate of $\delta_\star$, by considering the function

$$\delta^\star : \mathcal{P}(\mathbf{N}^+) \to \mathbf{R} : X \mapsto 1 - \delta_\star(\mathbf{N}^+ \setminus X), \tag{1}$$

see [8, § 2], and it is relatively simple to show that, for all $X, Y \subseteq \mathbf{N}^+$, the following elementary properties hold, see [8, Proposition 2.1]:

(L5) $\delta_\star(X) \le \delta_\star(Y)$ and $\delta^\star(X) \le \delta^\star(Y)$ if $X \subseteq Y$;
(L6) $\delta^\star(X \cup Y) \le \delta^\star(X) + \delta^\star(Y)$;
(L7) $\delta^\star(X) = \delta^\star(Y)$ provided that $|X \triangle Y| < \infty$;
(L8) $\delta_\star(X) \le \delta^\star(X)$;
(L9) $\delta_\star(\varnothing) = \delta^\star(\varnothing) = 0$ and $\delta^\star(\mathbf{N}^+) = 1$.

Remarkably, axioms (L1)-(L9) are all satisfied by letting $\delta_\star$ and $\delta^\star$ be, respectively, the lower and upper asymptotic densities (on $\mathbf{N}^+$), see [8, Proposition 3.1] and Example 4 below, and the same is true if the two asymptotic densities are replaced, respectively, by the lower and upper Banach densities, see [8, p. 299].

The basic goal of this paper is actually to give an axiomatization of the notion of upper (and lower) density, which is "smoother" than Freedman and Sember's, as it implies some desirable properties that do not necessarily hold true for a function subjected to conditions (L1)-(L4) and for its conjugate, see Examples 1 and 7 below.

Similar goals have been pursued by several authors in the past, though to the best of our knowledge earlier work on the subject has been mostly focused on the investigation of densities raising as a limit (in a broad sense) of a sequence or a net of measures, see, e.g., R. C. Buck [5], R. Alexander [1], D. Maharam [25], T. Šalát and R. Tijdeman [39], A. H. Mekler [27], A. Fuchs and R. Giuliano Antonini [9], A. Blass, R. Frankiewicz, G. Plebanek, and C. Ryll-Nardzewski [2], M. Sleziak and M. Ziman [42], and M. Di Nasso [6].

Plan of the paper. In § 1, we fix basic notation and terminology. In § 2, we introduce five axioms we use to shape the notions of upper and lower density studied in this work, along with the related notion of induced density. In §§ 3-5, we establish the mutual independence of our axioms, provide examples of functions that are, or are not, upper or lower densities (in the sense of our definitions), and prove (Theorem 2) that the range of an induced density is the interval $[0, 1]$, respectively. In § 6, we contrast our axioms with those of Freedman and Sember and derive some "structural properties". Most notably, we show that the set of all upper densities is convex, has an explicitly identified element (i.e., the upper Buck density, as defined by equation

(8)) that is both maximum and extremal, and is closed in the topology of pointwise convergence on the set of all functions $\mathcal{P}(\mathbf{N}^+) \to \mathbf{R}$ (from which we obtain, see Example 8, that also the upper Pólya density is an upper density). Lastly in § 7, we draw a list of open questions.

Notation and conventions. We let $\mathbf{N}$ denote the the set of nonnegative integers (so, $0 \in \mathbf{N}$). For $a, b \in \mathbf{R} \cup \{\infty\}$ we write $[\![a, b]\!]$ for the discrete interval $[a, b] \cap \mathbf{Z}$, and for $X \subseteq \mathbf{R}$ we set $X^+ := X \cap]0, \infty[$. Also, we define $\mathbf{R}_0^+ := [0, \infty[$.

We let $\mathbf{H}$ be either $\mathbf{Z}$, $\mathbf{N}$, or $\mathbf{N}^+$: If, on the one hand, it makes no substantial difference to stick to the assumption that $\mathbf{H} = \mathbf{N}$ for most of the time, on the other, some statements will be sensitive to the actual choice of $\mathbf{H}$ (see, e.g., Example 5 or Question 6).

Unless differently specified, the letters h, i, j, k, and l, with or without subscripts, will stand for nonnegative integers, the letters m and n for positive integers, the letter p for a positive (rational) prime, and the letter s for a positive real number.

Given $X \subseteq \mathbf{R}$ and $h, k \in \mathbf{R}$, we let $k \cdot X + \{h\} := \{kx + h : x \in X\}$ and take an arithmetic progression of $\mathbf{H}$ to be any set of the form $k \cdot \mathbf{H} + \{h\}$ with $k \in \mathbf{H} \setminus \{0\}$ and $h \in \mathbf{H} \cup \{0\}$; we will write $k \cdot X + h$ in place of $k \cdot X + \{h\}$ if there is no risk of ambiguity. We emphasize that, in this work, *arithmetic progressions are always infinite, unless noted otherwise*.

For $X, Y \subseteq \mathbf{H}$, we define $X^c := \mathbf{H} \setminus X$ and $X \triangle Y := (X \setminus Y) \cup (Y \setminus X)$. Furthermore, we say that a sequence $(x_n)_{n \ge 1}$ is the natural enumeration of a set $X \subseteq \mathbf{N}$ if $X = \{x_n : n \in \mathbf{N}^+\}$ and $x_n < x_{n+1}$ for each $n \in \mathbf{N}^+$. For a set S we let $\mathcal{P}(S)$ be the power set of S.

Lastly, given a partial function f from a set X to a set Y, herein denoted by $f : X \nrightarrow Y$, we write $\operatorname{dom}(f)$ for its domain, i.e., the set of all $x \in X$ such that $y = f(x)$ for some $y \in Y$.

Further notations and terminology, if not explained when first introduced, are standard or should be clear from the context.

2. Upper and lower densities (and quasi-densities)

We write $\mathsf{d}_\star$ and $\mathsf{d}^\star$, respectively, for the functions $\mathcal{P}(\mathbf{H}) \to \mathbf{R}$ mapping a set $X \subseteq \mathbf{H}$ to its lower and upper asymptotic (or natural) density, i.e.,

$$\mathsf{d}_\star(X) := \liminf_{n\to\infty} \frac{|X \cap [1, n]|}{n} \quad \text{and} \quad \mathsf{d}^\star(X) := \limsup_{n\to\infty} \frac{|X \cap [1, n]|}{n}.$$

Moreover, we denote by $\mathsf{bd}_\star$ and $\mathsf{bd}^\star$, respectively, the functions $\mathcal{P}(\mathbf{H}) \to \mathbf{R}$ taking a set $X \subseteq \mathbf{H}$ to its lower and upper Banach (or uniform) density, viz.

$$\mathsf{bd}_\star(X) := \lim_{n\to\infty} \min_{l\ge 0} \frac{|X \cap [l+1, l+n]|}{n} \quad \text{and} \quad \mathsf{bd}^\star(X) := \lim_{n\to\infty} \max_{l\ge 0} \frac{|X \cap [l+1, l+n]|}{n}.$$

The existence of the latter limits, as well as equivalent definitions of $\mathsf{bd}_\star$ and $\mathsf{bd}^\star$ are discussed, e.g., in [18]. Note that, although the set X may contain zero or negative integers, the definitions above only involve the positive part of X, cf. [19, p. xvii] and Example 4 below.

Now, it is well known, see, e.g., [29, § 2] and [30, Theorem 11.1], that if X is an infinite subset of $\mathbf{N}^+$ and $(x_n)_{n\ge 1}$ is the natural enumeration of X, then

$$\mathsf{d}_\star(X) = \liminf_{n\to\infty} \frac{n}{x_n} \quad \text{and} \quad \mathsf{d}^\star(X) = \limsup_{n\to\infty} \frac{n}{x_n}. \tag{2}$$

Since $\mathsf{d}_\star(X) = \mathsf{d}^\star(X) = 0$ for every finite $X \subseteq \mathbf{N}^+$, it follows that $\mathsf{d}_\star(k \cdot X + h) = \frac{1}{k}\mathsf{d}_\star(X)$ and $\mathsf{d}^\star(k \cdot X + h) = \frac{1}{k}\mathsf{d}^\star(X)$ for all $X \subseteq \mathbf{H}$, $h \in \mathbf{N}$, and $k \in \mathbf{N}^+$. This is often considered as one of the most desirable features a density ought to have, cf. [16, § 3]. One reason is that, roughly speaking, a density on $\mathbf{N}^+$ should *ideally* approximate a shift-invariant probability measure with the further property that the measure of $k \cdot \mathbf{N}^+$ is $\frac{1}{k}$ for all $k \in \mathbf{N}^+$. But no *countably* additive probability measure with this latter property exists, see [46, Chapter III.1, Theorem 1], and it is known that the existence of *finitely* additive, shift-invariant probability measures $\mathcal{P}(\mathbf{N}^+) \to \mathbf{R}$ cannot be proved in the frame of classical mathematics without (some form of) the axiom of choice (cf. Remark 3), which makes them unwieldy in many practical situations. On the other hand, it is straightforward that, for every $X \subseteq \mathbf{N}^+$, $h \in \mathbf{N}$, and $k \in \mathbf{N}^+$,

$$\min_{l \ge 0} \left|(k \cdot X + h) \cap [l+1, l+nk]\right| = \min_{l \ge 0} \left|X \cap [l+1, l+n]\right| + O(1) \quad (n \to \infty),$$

which implies $\mathsf{bd}_\star(k \cdot X + h) = \frac{1}{k}\mathsf{bd}_\star(X)$. And the same holds for $\mathsf{bd}^\star$, by similar arguments.

In the wake of [8], it thus seems natural to axiomatize a notion of upper density by considering a function $\mu^\star : \mathcal{P}(\mathbf{H}) \to \mathbf{R}$ subjected to the following axioms, the first three of which correspond, respectively, to conditions (L4), (L5) and (L6) above (see Example 7, and the comments introducing it in § 6, for a more detailed comparison of the two axiomatizations):

(F1) $\mu^\star(\mathbf{H}) = 1$;
(F2) $\mu^\star(X) \le \mu^\star(Y)$ for all $X, Y \subseteq \mathbf{H}$ with $X \subseteq Y$;
(F3) $\mu^\star(X \cup Y) \le \mu^\star(X) + \mu^\star(Y)$ for all $X, Y \subseteq \mathbf{H}$;
(F4) $\mu^\star(k \cdot X) = \frac{1}{k}\mu^\star(X)$ for all $X \subseteq \mathbf{H}$ and $k \in \mathbf{N}^+$;
(F5) $\mu^\star(X + h) = \mu^\star(X)$ for all $X \subseteq \mathbf{H}$ and $h \in \mathbf{N}$.

To ease the exposition, we will say that a function $\mu^\star : \mathcal{P}(\mathbf{H}) \to \mathbf{R}$ is: monotone if it satisfies (F2); subadditive if it satisfies (F3); (finitely) additive if $\mu^\star(X \cup Y) = \mu^\star(X) + \mu^\star(Y)$ whenever $X, Y \subseteq \mathbf{H}$ and $X \cap Y = \varnothing$; (-1)-homogeneous if it satisfies (F4); and translational invariant (or shift-invariant, or translational symmetric) if it satisfies (F5).

For future reference, observe that conditions (F4) and (F5) together are equivalent to:

(F6) $\mu^\star(k \cdot X + h) = \frac{1}{k}\mu^\star(X)$ for all $X \subseteq \mathbf{H}$ and $h, k \in \mathbf{N}^+$.

Note that (F1) and (F2) together imply that $\mu^\star(X) \le 1$ for every $X \subseteq \mathbf{H}$, which is, however, false if (F2) is not assumed, see Theorem 1.

With this in mind, let $\mu^\star$ be a function $\mathcal{P}(\mathbf{H}) \to \mathbf{R}$, and similarly to (1) consider the map

$$\mu_\star : \mathcal{P}(\mathbf{H}) \to \mathbf{R} : X \mapsto 1 - \mu^\star(X^c).$$

We call $\mu_\star$ the *dual*, or *conjugate*, of $\mu^\star$; notice that $\mu^\star$ is self-dual, in the sense that the dual of the dual of $\mu^\star$ is $\mu^\star$ itself (namely, $\mu^\star$ is the dual of $\mu_\star$). Moreover, we say that $(\mu_\star, \mu^\star)$ is a *conjugate pair* (on $\mathbf{H}$) if $\mu^\star(\mathbf{H}) = 1$, and we refer to $\mu_\star$ as the *lower dual* of $\mu^\star$, and reciprocally to $\mu^\star$ as the *upper dual* of $\mu_\star$, if $\mu_\star(X) \le \mu^\star(X)$ for every $X \subseteq \mathbf{H}$; this happens, in particular, if $\mu^\star$ is subadditive and $(\mu_\star, \mu^\star)$ is a conjugate pair, see Proposition 2(vi) below.

Moreover, we say that $\mu^\star$ is an *upper quasi-density* (on $\mathbf{H}$) if it satisfies (F1), (F3) and (F6) and, in addition, $\mu^\star(X) \le 1$ for every $X \subseteq \mathbf{H}$, in which case, based on Proposition 2(vi), we refer to $\mu_\star$ as a *lower quasi-density* (on $\mathbf{H}$), or more precisely as the lower quasi-density associated

with $\mu^\star$. In particular, we refer to $\mu^\star$ as an *upper density* if it is a monotone upper quasi-density, and then $\mu_\star$ is called a *lower density*, or more specifically the lower density associated with $\mu^\star$.

It follows that $\mathsf{d}^\star$ and $\mathsf{bd}^\star$ are both upper densities in the sense of the above definitions, and their lower duals are, respectively, $\mathsf{d}_\star$ and $\mathsf{bd}_\star$; more examples will be given in § 4.

The proofs of the following two propositions are left as an exercise for the interested reader.

Proposition 1. *Let $\mu^\star$ be a monotone and subadditive function $\mathcal{P}(\mathbf{H}) \to \mathbf{R}$, and let $X, Y \subseteq \mathbf{H}$.*

(i) *If $\mu^\star(X \triangle Y) = 0$, then $\mu^\star(X) = \mu^\star(Y)$ and $\mu^\star(X^c) = \mu^\star(Y^c)$.*
(ii) *If $\mu^\star(Y) = 0$, then $\mu^\star(X) = \mu^\star(X \cup Y) = \mu^\star(X \setminus Y)$.*
(iii) *If $\mu^\star(X \setminus Y) = 0$, then $\mu^\star(X \cap Y) = \mu^\star(X)$.*
(iv) *If $\mu^\star(X) < \mu^\star(Y)$, then $0 < \mu^\star(Y) - \mu^\star(X) \le \mu^\star(Y \setminus X)$.*

Proposition 2. *Let $(\mu_\star, \mu^\star)$ be a conjugate pair on $\mathbf{H}$. The following hold:*

(i) *$\mu_\star(\varnothing) = 0$, and $\mathrm{Im}(\mu^\star) \subseteq [0,1]$ if and only if $\mathrm{Im}(\mu_\star) \subseteq [0,1]$.*
(ii) *If $\mu^\star$ is subadditive and $X_1, \ldots, X_n \subseteq \mathbf{H}$, then $\mu^\star(X_1 \cup \cdots \cup X_n) \le \sum_{i=1}^n \mu^\star(X_i)$.*
(iii) *Let $\mu^\star$ be (-1)-homogeneous. Then $\mu^\star(\varnothing) = 0$ and $\mu_\star(\mathbf{H}) = 1$. In addition, $\mu^\star(\{0\}) = 0$ provided that $0 \in \mathbf{H}$.*

Moreover, for all $X, Y \subseteq \mathbf{H}$ we have the following:

(iv) *Assume $\mu^\star$ is monotone. If $X \subseteq Y$, then $\mu_\star(X) \le \mu_\star(Y)$.*
(v) *If $\mu^\star(X \triangle Y) = 0$ and $\mu^\star$ satisfies (F2)-(F3), then $\mu_\star(X) = \mu_\star(Y)$ and $\mu_\star(X^c) = \mu_\star(Y^c)$.*
(vi) *If $\mu^\star$ is subadditive, then $\max(\mu_\star(X), 0) \le \mu^\star(X)$ and $\mu_\star(X) + \mu_\star(Y) \le 1 + \mu_\star(X \cap Y)$.*

A few examples and remarks are now in order. In particular, the first example is borrowed from [8, p. 297], while the second shows that conditions (F1)-(F4) alone are not even enough to guarantee that, for a function $\mu^\star : \mathcal{P}(\mathbf{H}) \to \mathbf{R}$, $\mu^\star(X) = 0$ whenever $X \subseteq \mathbf{H}$ is finite.

Example 1. Let $\delta_\star$ be the function $\mathcal{P}(\mathbf{H}) \to \mathbf{R}$ sending a set $X \subseteq \mathbf{H}$ either to 1 if $|X^c| < \infty$ or to 0 otherwise. It is found that $\delta_\star$ satisfies axioms (L1)-(L4) and $\delta_\star(k \cdot \mathbf{H}) = 0 \ne \frac{1}{k}\delta_\star(\mathbf{H})$ for $k \ge 2$; also, if $\delta^\star$ is the conjugate of $\delta_\star$, then $\delta^\star(k \cdot X) = \delta^\star(X) = 1$ for every $k \ge 1$ and finite $X \subseteq \mathbf{H}$. To wit, neither $\delta_\star$ nor $\delta^\star$ are (-1)-homogeneous.

Example 2. A "lower density" need not be subadditive (cf. Remark 3 below). We know, in fact, from the introduction and the considerations at the beginning of this section that $\mathsf{d}^\star$ and $\mathsf{bd}^\star$ satisfy (F1)-(F4), and $\mathsf{d}_\star$ and $\mathsf{bd}_\star$ satisfy (F1), (F2) and (F4). Yet, if we let

$$X := \bigcup_{n \ge 1} [\![(2n-1)!, (2n)! - 1]\!] \subseteq \mathbf{N}^+ \quad \text{and} \quad Y := \bigcup_{n \ge 1} [\![(2n)!, (2n+1)! - 1]\!] \subseteq \mathbf{N}^+,$$

then $\mathsf{d}_\star(X) = \mathsf{d}_\star(Y) = \mathsf{bd}_\star(X) = \mathsf{bd}_\star(Y) = 0$, as is easily proved, e.g., by Lemma 1 below and the fact that $0 \le \mathsf{bd}_\star(Z) \le \mathsf{d}_\star(Z)$ for all $Z \subseteq \mathbf{H}$. But $\mathsf{d}_\star(X \cup Y) = \mathsf{bd}_\star(X \cup Y) = 1$, because $X \cup Y = \mathbf{N}^+$, and hence we see that neither $\mathsf{d}_\star$ nor $\mathsf{bd}_\star$ are subadditive.

Example 3. It is straightforward to check (we omit the details) that the function

$$\mathfrak{m} : \mathcal{P}(\mathbf{H}) \to \mathbf{R} : X \mapsto \frac{1}{\inf(X^+)},$$

where $\inf(\varnothing) := \infty$ and $\frac{1}{\infty} := 0$, satisfies conditions (F1)-(F4). Note that $\mathfrak{m}(X + h) \ne \mathfrak{m}(X) \ne 0$ for every $X \subseteq \mathbf{H}$ and $h \in \mathbf{N}^+$ such that $X^+ \ne \varnothing$.

Remark 1. Proposition 2(vi) yields that an upper quasi-density is necessarily nonnegative, hence its range is contained in $[0,1]$.

Remark 2. Four out of the five axioms we are using to shape our notion of upper density are essentially the same as four out of the seven axioms considered in [16] as a *suggestion* for an abstract notion of density on $\mathbf{N}^+$, see in particular axioms (A2)-(A5) in [16, § 3].

Remark 3. All the results in the present paper can be proved by appealing, e.g., to the usual device of Zermelo-Fraenkel set theory *without* the axiom of choice (ZF for short). However, if we assume to work in Zermelo-Fraenkel set theory *with* the axiom of choice (ZFC for short), then it follows from [7, Appendix 5C] that there are uncountably many nonnegative additive functions $\theta : \mathcal{P}(\mathbf{N}^+) \to \mathbf{R}$ such that $\theta(\mathbf{N}^+) = 1$ and $\theta(k \cdot X + h) = \frac{1}{k}\theta(X)$ for all $X \subseteq \mathbf{N}^+$ and $h, k \in \mathbf{N}^+$. In particular, the function $\mu^\star : \mathcal{P}(\mathbf{H}) \to \mathbf{R} : X \mapsto \theta(X^+)$ is an additive upper density on $\mathbf{H}$.

On the other hand, we have from Proposition 6 below that if $\mu^\star$ is an upper quasi-density on $\mathbf{H}$ then $\mu^\star(X) = 0$ for every finite $X \subseteq \mathbf{H}$, and it is provable in ZF that the existence of an additive measure $\theta : \mathcal{P}(\mathbf{N}^+) \to \mathbf{R}$ which vanishes on singletons yields the existence of a subset of $\mathbf{R}$ without the property of Baire, see [40, §§ 29.37 and 29.38]. Yet, ZF alone does not prove the existence of such an additive measure θ, see [34].

So, putting it all together, the existence of an additive upper quasi-density on $\mathbf{H}$ is provable in ZFC, but independent of ZF.

Remark 4. Axiom (F6) is incompatible with the following condition, which is often referred to as sigma subadditivity (in analogy to the sigma additivity of measures):

(F7) If $(X_n)_{n\ge1}$ is a sequence of subsets of $\mathbf{H}$, then $\mu^\star\Big(\bigcup_{n\ge1} X_n\Big) \le \sum_{n\ge1} \mu^\star(X_n)$.

In fact, let $(q_n)_{n\ge1}$ be a (strictly) increasing sequence of positive integers for which $\frac{1}{2} > \sum_{n\ge1} \frac{1}{q_n}$, and for each n set $X_n := q_n \cdot \mathbf{N} + n$ if $\mathbf{H} = \mathbf{N}^+$ and $X_n := (q_n \cdot \mathbf{H} + n - 1) \cup (q_n \cdot \mathbf{H} + q_n - n)$ otherwise. By construction, we have $\mathbf{H} = \bigcup_{n\ge1} X_n$. Thus, if a function $\mu^\star : \mathcal{P}(\mathbf{H}) \to \mathbf{R}$ satisfies (F1) and (F6) then $\mu^\star(\mathbf{H}) = 1 > \sum_{n\ge1} \mu^\star(X_n)$, which is not compatible with (F7).

To conclude this section, let $(\mu_\star, \mu^\star)$ be a conjugate pair and μ the partial function $\mathcal{P}(\mathbf{H}) \nrightarrow \mathbf{R} : X \mapsto \mu^\star(X)$ whose domain is the set $\{X \subseteq \mathbf{H} : \mu_\star(X) = \mu^\star(X)\}$. Equivalently, we have

$$\mathrm{dom}(\mu) = \{X \in \mathcal{P}(\mathbf{H}) : \mu^\star(X) + \mu^\star(X^c) = 1\} = \{X \in \mathcal{P}(\mathbf{H}) : \mu_\star(X) + \mu_\star(X^c) = 1\},$$

so $\mathrm{dom}(\mu)$ is closed under complementation, and by Proposition 2(vi) we have that $X \in \mathrm{dom}(\mu)$ whenever $\mu^\star$ is monotone and subadditive and $X \subseteq \mathbf{H}$ is such that $\mu^\star(X) = 0$ or $\mu_\star(X) = 1$. If, in particular, $\mu^\star$ is an upper quasi-density (respectively, upper density), we refer to μ as the *quasi-density* (respectively, *density*) induced by $\mu^\star$.

More specifically, the density d induced by $\mathsf{d}^\star$ is called the asymptotic (or natural) density (on $\mathbf{H}$), and for every $X \in \mathrm{dom}(\mathsf{d})$ we have $\mathsf{d}(X) = \lim_{n\to\infty} \frac{1}{n}|X \cap [\![1, n]\!]|$, cf. [19, p. xvii].

In fact, many authors have investigated properties of μ and $\mathrm{dom}(\mu)$, especially in the case when $\mu^\star$ is either the upper asymptotic density (on $\mathbf{N}^+$), see, e.g., [5], [39], [25], [27], [2], [42, 43], and [6], or the upper Buck density (on $\mathbf{N}^+$), see, e.g., [4], [31, §§ 3 and 4], [32], and [33].

3. Independence of the axioms

The independence of (F1) from the other axioms is obvious, while that of (F3) and (F5) follows from Examples 2 and 3, respectively. Moreover, (F4) is independent from (F1)-(F3) and (F5): Indeed, these latter conditions are all satisfied by the constant function $\mathcal{P}(\mathbf{H}) \to \mathbf{R} : X \mapsto 1$, which however does not satisfy (F4). In contrast, the independence of (F2) from (F1), (F3) and (F6) is much more delicate, and it clearly follows from the existence of an upper quasi-density that is not an upper density, which is what we are going to prove.

We begin with a rather simple lemma on the upper (and the lower) asymptotic density of subsets of $\mathbf{N}^+$ in which "large gaps" alternate to "large intervals" (we omit the proof).

Lemma 1. *Let $(a_n)_{n\ge1}$ and $(b_n)_{n\ge1}$ be sequences of positive real numbers such that:*

(i) $a_n + 1 \le b_n < a_{n+1}$ *for all sufficiently large n;*
(ii) $a_n/b_n \to \ell$ *as $n \to \infty$;*
(iii) $b_n/a_{n+1} \to 0$ *as $n \to \infty$.*

In addition, pick $k \in \mathbf{N}^+$ and $h \in \mathbf{Z}$, and let $X \subseteq \mathbf{N}^+$ be the intersection of the sets $\bigcup_{n\ge1} [\![a_n + 1, b_n]\!]$ and $k \cdot \mathbf{H} + h$. Then $\mathsf{d}^\star(X) = \frac{1}{k}(1-\ell)$ and $\mathsf{d}_\star(X) = 0$; in particular, if $\alpha \in [0,1]$ and we let $a_n := \alpha(2n-1)! + (1-\alpha)(2n)!$ and $b_n := (2n)! + 1$, then $\mathsf{d}^\star(X) = \frac{\alpha}{k}$ and $\mathsf{d}_\star(X) = 0$.

Incidentally, a byproduct of Lemma 1 is that $\mathrm{Im}(\mathsf{d}^\star) = [0,1]$, which the reader may want to compare with Theorem 2 below, where we prove a stronger version of the same result.

The next step is now to show how the independence of (F2) can be drawn from the existence of a suitable "indexing" $\iota : \mathcal{P}(\mathbf{H}) \to \mathbf{N}^+ \cup \{\infty\}$ associated with an upper quasi-density $\mu^\star$ on $\mathbf{Z}$, which may or may not satisfy (F2), such that $\mu^\star(\mathbf{H}) = 1$.

More specifically, we will prove that the existence of such an indexing can be used to construct infinitely many non-monotone functions $\mu^\star : \mathcal{P}(\mathbf{H}) \to \mathbf{R}$ that, on the one hand, are not upper densities (in fact, we will show that they can even be unbounded), and on the other, are subjected to some additional constraints on the values they can attain: This implies, in particular, that (F2) is independent of (F1), (F3) and (F6) also in the case where $\mu^\star$ is an upper quasi-density.

Lemma 2. *Let $\mu^\star$ be an upper quasi-density on $\mathbf{Z}$ with the property that $\mu^\star(\mathbf{H}) = 1$. Suppose there exists a function $\iota : \mathcal{P}(\mathbf{H}) \to \mathbf{N}^+ \cup \{\infty\}$ such that:*

(I1) $\iota(\mathbf{H}) = 1$;
(I2) $\iota(X) = \infty$ *for some $X \subseteq \mathbf{H}$ only if $\mu^\star(X) = 0$;*
(I3) $\iota(Y) \le \iota(X)$ *whenever $X \subseteq Y \subseteq \mathbf{H}$ and $\mu^\star(Y) > 0$;*
(I4) $\iota(X) = \iota(k \cdot X + h)$ *for all $X \subseteq \mathbf{H}$ and $h, k \in \mathbf{N}^+$ with $\mu^\star(X) > 0$.*

Let $(a_n)_{n\ge1}$ be a nondecreasing (real) sequence with $a_1 = 1$, and consider the function

$$\theta^\star : \mathcal{P}(\mathbf{H}) \to \mathbf{R} : X \mapsto \begin{cases} a_{\iota(X)}\mu^\star(X) & \text{if } \iota(X) < \infty \\ 0 & \text{otherwise} \end{cases}.$$

Then $\theta^\star$ satisfies axioms (F1), (F3) and (F6). Now let $\varrho_n := \sup\{\mu^\star(X) : X \subseteq \mathbf{H}, \iota(X) = n\}$ and, in addition to the previous assumptions, suppose that:

(I5) $0 < \varrho_{n+1} \le \varrho_n$ *for every n, and $\varrho_n < 1$ for every sufficiently large n;*

(I6) *No matter how we choose $n \in \mathbf{N}^+$ and $\varepsilon \in \mathbf{R}^+$, there exist $X, Y \subseteq \mathbf{H}$ such that $Y \subsetneq X$, $\iota(X) = n$, $\iota(Y) = n+1$, and $\varrho_{n+1} \le (1+\varepsilon)\mu^\star(Y)$.*

Lastly, let $K \in [1, \infty]$. It is then possible to choose the "weights" $a_2, a_3, \ldots$ in a way that $\theta^\star$ is not monotone and $\sup_{X \in \mathcal{P}(\mathbf{H})} \theta^\star(X) = K$.

Proof. First we note that $\theta^\star$ is nonnegative. Next, it is immediate that $\theta^\star$ satisfies (F1), because by (I1) and the hypothesis that $\mu^\star(\mathbf{H}) = a_1 = 1$ we have $\theta^\star(\mathbf{H}) = a_1\mu^\star(\mathbf{H}) = 1$.

As for (F3), pick $X, Y \subseteq \mathbf{H}$, and assume without loss of generality that $\mu^\star(X \cup Y) > 0$ and $\mu^\star(Y) \le \mu^\star(X)$. Then we have from (I2) and the fact that $\mu^\star$ is subadditive that $\iota(X) < \infty$ (in particular, $\iota(X) = \infty$ would entail that $\mu^\star(X \cup Y) \le 2\mu^\star(X) = 0$, namely, a contradiction), which ultimately implies from (I3) that $\iota(X \cup Y) \le \iota(X) < \infty$. Thus,

$$\begin{aligned}\theta^\star(X \cup Y) &= a_{\iota(X\cup Y)}\mu^\star(X \cup Y) \le a_{\iota(X\cup Y)}\mu^\star(X) + a_{\iota(X\cup Y)}\mu^\star(Y) \\ &\le a_{\iota(X)}\mu^\star(X) + a_{\iota(X\cup Y)}\mu^\star(Y) \le \theta^\star(X) + \theta^\star(Y),\end{aligned} \tag{3}$$

where we have used, among other things, that $a_n \le a_{n+1}$ for all n. To wit, also $\theta^\star$ satisfies (F3).

Lastly, fix $X \subseteq \mathbf{H}$ and $h, k \in \mathbf{N}^+$. We want to show that $\theta^\star(k \cdot X + h) = \theta^\star(X)$. This is trivial if $\mu^\star(X) = 0$, as $\mu^\star$ being an upper quasi-density implies from (F6) that $\mu^\star(k \cdot X + h) = \frac{1}{k}\mu^\star(X) = 0$, and of course $\theta^\star(Y) = 0$ for every $Y \subseteq \mathbf{H}$ with $\mu^\star(Y) = 0$. So assume $\mu^\star(X) \ne 0$. Then $\iota(k \cdot X + h) = \iota(X) < \infty$ by (I2) and (I4), with the result that

$$\theta^\star(k \cdot X + h) = a_{\iota(k\cdot X+h)}\mu^\star(k \cdot X + h) = \frac{1}{k}a_{\iota(X)}\mu^\star(X) = \frac{1}{k}\theta^\star(X),$$

where we have used again that $\mu^\star$ satisfies (F6). It follows that also $\theta^\star$ satisfies (F6), which, together with the rest, proves the first part of the lemma.

As for the second part, fix $K \in [1, \infty]$ and let ι satisfy (I5) and (I6). Then we have from (I1) and $\mu^\star$ being an upper quasi-density that there is an integer $v \ge 2$ such that, for $n \ge v$,

$$0 < \varrho_{n+1} \le \varrho_n < \varrho_{v-1} = \cdots = \varrho_1 = 1. \tag{4}$$

So we find it natural to distinguish between two cases according to the actual value of K.

Case 1: $1 < K \le \infty$. Based on (I5), we make $(a_n)_{n\ge1}$ into a nondecreasing sequence by letting $a_n := \varrho_n^{-1}\min(2^{n-1}, K)$; observe that $a_1 = 1$.

Then, given $X \subseteq \mathbf{H}$ with $\iota(X) < \infty$, we have $\theta^\star(X) = a_{\iota(X)}\mu^\star(X) \le \min(2^{\iota(X)-1}, K) \le K$. In addition, if we set $\delta := \min(2, K)$ and pick $\varepsilon \in]0, 1 - \delta^{-1}[$, then (by definition of the supremum) we can find $X \subseteq \mathbf{H}$ such that $\iota(X) = 2$ and $\mu^\star(X) \ge (1-\varepsilon)\varrho_2$, so that

$$\theta^\star(X) = a_2\mu^\star(X) \ge (1-\varepsilon)\delta > 1 = \theta^\star(\mathbf{H}).$$

Consequently, we see that $\sup_{X\in\mathcal{P}(\mathbf{H})}\theta^\star(X) = K$ and $\theta^\star$ is non-monotone.

Case 2: $K = 1$. Similarly to the previous case, but now based on (4), we make $(a_n)_{n\ge1}$ into a nondecreasing sequence by taking $a_1 := \cdots := a_v := 1$ and $a_n := \frac{1}{2}(1+\varrho_v)\varrho_n^{-1}$ for $n \ge v+1$.

Then let $X \subseteq \mathbf{H}$ be such that $n := \iota(X) < \infty$, so that $\theta^\star(X) = a_n\mu^\star(X)$. If $1 \le n \le v$, we have $\theta^\star(X) = \mu^\star(X) \in [0, 1]$; otherwise,

$$\theta^\star(X) = \frac{1}{2}(1+\varrho_v)\varrho_n^{-1}\mu^\star(X) \le \frac{1}{2}(1+\varrho_v) \le 1,$$

since $\iota(X) = n$ yields that $\mu^\star(X) \le \varrho_n$. It follows that $\theta^\star(X) \le 1 = \theta^\star(\mathbf{H})$ for every $X \subseteq \mathbf{H}$, namely, $\sup_{X \in \mathcal{P}(\mathbf{H})} \theta^\star(X) = K = 1$. So we are left to show that $\theta^\star$ is non-monotone.

For this, note that $\varrho_v < \frac{1}{2}(1+\varrho_v)$, and let $\varepsilon \in \mathbf{R}^+$ be such that $\varrho_v < \frac{1}{2}(1+\varrho_v)(1+\varepsilon)^{-1}$. By (I6), there then exist $X, Y \subseteq \mathbf{H}$ with the property that $Y \subsetneq X$, $\iota(X) = v$, $\iota(Y) = v+1$, and $\varrho_{v+1} \le (1+\varepsilon)\mu^\star(Y)$, from which we see that $\theta^\star$ is non-monotone, because

$$\theta^\star(X) = a_v \mu^\star(X) = \mu^\star(X) \le \varrho_v < \frac{1}{2}(1+\varrho_v)(1+\varepsilon)^{-1} \le a_{v+1}\mu^\star(Y) = \theta^\star(Y).$$

Putting it all together, the proof is thus complete. ∎

Now we prove that there exists a function $\iota : \mathcal{P}(\mathbf{H}) \to \mathbf{N}^+ \cup \{\infty\}$ that fulfills the assumptions of Lemma 2. This is the content of the following:

Lemma 3. *Let $\mu^\star$ be the upper asymptotic density on $\mathbf{Z}$ and ι the function $\mathcal{P}(\mathbf{H}) \to \mathbf{N}^+ \cup \{\infty\}$ taking a set $X \subseteq \mathbf{H}$ to the infimum of the integers $n \ge 1$ for which there exists $Y \subseteq \mathbf{H}$ such that $\mu^\star(Y) \ge \frac{1}{n}$ and $|(q \cdot Y + r) \setminus X| < \infty$ for some $q \in \mathbf{N}^+$ and $r \in \mathbf{Z}$, with the convention that* $\inf(\varnothing) := \infty$. *Then ι satisfies conditions (I1)-(I6).*

Proof. To start with, observe that $\iota(X) \le \inf\{n \in \mathbf{N}^+ : \mu^\star(X) \ge 1/n\}$ for every $X \subseteq \mathbf{H}$, since $X = 1 \cdot X + 0$ and $|X \setminus X| = 0 < \infty$. This implies $\iota(\mathbf{H}) = 1$, and shows as well that $\iota(X) = \infty$ for some $X \subseteq \mathbf{H}$ only if $\mu^\star(X) = 0$. To wit, ι satisfies (I1) and (I2).

Next, let $X \subseteq Y \subseteq \mathbf{H}$. We claim that $\iota(Y) \le \iota(X)$. This is obvious if $\iota(X) = \infty$, so assume $\iota(X) < \infty$. Then, there exists $\mathcal{W} \subseteq \mathbf{H}$ such that $\mu^\star(\mathcal{W}) \ge 1/\iota(X)$ and $|(q \cdot \mathcal{W} + r) \setminus X| < \infty$ for some $q \in \mathbf{N}^+$ and $r \in \mathbf{Z}$. So we have $|(q \cdot \mathcal{W} + r) \setminus Y| < \infty$, because $S \setminus Y \subseteq S \setminus X$ for every $S \subseteq \mathbf{H}$ (by the fact that $X \subseteq Y$). This proves that $\iota(Y) \le \iota(X)$, and hence, by the arbitrariness of X and Y, we find that ι satisfies (I3).

Lastly, we come to (I4). Let $X \subseteq \mathbf{H}$ be such that $\mu^\star(X) > 0$, and pick $h, k \in \mathbf{N}^+$. Then it follows from (I2) that $\iota(X) < \infty$ and $\iota(k \cdot X + h) < \infty$, because $\mu^\star(k \cdot X + h) = \frac{1}{k}\mu^\star(X) \ne 0$ by the fact that $\mu^\star$ satisfies (F6). We want to show that $\iota(X) = \iota(k \cdot X + h)$.

To begin, there exists (by definition) a set $Y \subseteq \mathbf{H}$ such that $\mu^\star(Y) \ge 1/\iota(X)$ and $|(q \cdot Y + r) \setminus X| < \infty$ for some $q \in \mathbf{N}^+$ and $r \in \mathbf{Z}$. Therefore, we see that $|(\tilde{q} \cdot Y + \tilde{r}) \setminus (k \cdot X + h)| < \infty$ for $\tilde{q} := qk$ and $\tilde{r} := rk + h$, which in turn implies that $\iota(k \cdot X + h) \le \iota(X)$.

As for the reverse inequality, there exists (again by definition) a set $Y \subseteq \mathbf{H}$ such that $\mu^\star(Y) \ge 1/\iota(k \cdot X + h)$ and $|(q \cdot Y + r) \setminus (k \cdot X + h)| < \infty$ for some $q \in \mathbf{N}^+$ and $r \in \mathbf{Z}$.

Suppose to a contradiction that $\mu^\star(Y \cap (k \cdot \mathbf{H} + l)) < \frac{1}{k}\mu^\star(Y)$ for every $l \in [\![0, k-1]\!]$. Then we get by axioms (F3) and (F6) and Proposition 6 below that

$$\mu^\star(Y) \le \mu^\star\left(\bigcup_{l=0}^{k-1} (Y \cap (k \cdot \mathbf{H} + l))\right) \le \sum_{l=0}^{k-1} \mu^\star(Y \cap (k \cdot \mathbf{H} + l)) < \sum_{l=0}^{k-1} \frac{1}{k}\mu^\star(Y) = \mu^\star(Y),$$

a contradiction. Accordingly, pick $l \in [\![0, k-1]\!]$ with the property that $\mu^\star(Y \cap (k \cdot \mathbf{H} + l)) \ge \frac{1}{k}\mu^\star(Y)$, and let $Q \subseteq \mathbf{H}$ be such that $k \cdot Q + l = Y \cap (k \cdot \mathbf{H} + l) \subseteq Y$. We have

$$|(qk \cdot Q + lq + r) \setminus (k \cdot X + h)| \le |(q \cdot Y + r) \setminus (k \cdot X + h)| < \infty,$$

and this entails that for every $y \in Q \setminus S$, where $S \subseteq \mathbf{H}$ is a finite set, there is some $x \in X$ such that $qky + lq + r - h = kx$, with the result that $\tilde{r} := \frac{1}{k}(lq + r - h)$ is an integer and $qy + \tilde{r} = x$.

It follows that $|(q \cdot Q + \tilde{r}) \setminus X| < \infty$, which is enough to conclude that $\iota(X) \le \iota(k \cdot X + h)$, since $\frac{1}{k}\mu^\star(Q) = \mu^\star(k \cdot Q + l) \ge \frac{1}{k}\mu^\star(Y)$, and hence $\mu^\star(Q) \ge \mu^\star(Y) \ge 1/\iota(k \cdot X + h)$.

To summarize, we have so far established that ι satisfies (I1)-(I4), and we are left with (I5) and (I6). For this, let V_α be, for each $\alpha \in]0,1[$, the set

$$\bigcup_{n \ge 1} [\![\alpha(2n-1)! + (1-\alpha)(2n)! + 1, (2n)! + 1]\!], \tag{5}$$

and for every $n \in \mathbf{N}^+$ assume $\varrho_n := \{\mu^\star(X) : X \subseteq \mathbf{H}, \iota(X) = n\}$. It is clear that $\varrho_1 = 1$, and we want to show that $\varrho_n = \frac{1}{n-1}$ for $n \ge 2$. To this end, we prove the following:

CLAIM. Fix $n \ge 2$ and let $\alpha \in \left[\frac{1}{n}, \frac{1}{n-1}\right[$. Then $\mu^\star(V_\alpha) = \alpha$ and $\iota(V_\alpha) = n$.

Proof. If $Y \subseteq \mathbf{H}$ and $\mu^\star(Y) \ge \frac{1}{n-1}$, then $|(q \cdot Y + r) \setminus V_\alpha| = \infty$ for all $q \in \mathbf{N}^+$ and $r \in \mathbf{Z}$: In fact, we have from axiom (F6) and Lemma 1 that

$$\mu^\star(q \cdot Y + r) = \frac{1}{q}\mu^\star(Y) \ge \frac{1}{(n-1)q} > \frac{\alpha}{q} = \mu^\star(V_\alpha \cap (q \cdot \mathbf{N} + r)),$$

so we get by Proposition 1(iv) that

$$\mu^\star((q \cdot Y + r) \setminus V_\alpha) = \mu^\star((q \cdot Y + r) \setminus (V_\alpha \cap (q \cdot \mathbf{N} + r))) > 0,$$

which is enough to conclude that $(q \cdot Y + r) \setminus V_\alpha$ is an infinite set, see Proposition 6 below, and ultimately shows that $\iota(V_\alpha) \ge n$. On the other hand, a further application of Lemma 1 yields that $\mu^\star(V_\alpha) = \alpha \ge \frac{1}{n}$, and hence $\iota(V_\alpha) \le n$. So putting it all together, the claim is proved. ∎

The above claim implies that $\varrho_n = \frac{1}{n-1}$ for $n \ge 2$, since it shows that for every $\varepsilon \in \mathbf{R}^+$ we can find a set $X \subseteq \mathbf{H}$ such that $\mu^\star(X) \ge \frac{1}{n-1} - \varepsilon$ and $\iota(X) = n$, and on the other hand, it is straightforward that $\varrho_n \le \frac{1}{n-1}$, as otherwise there would exist a set $X \subseteq \mathbf{H}$ such that $\iota(X) = n$ and $\mu^\star(X) \ge \frac{1}{n-1}$, which is impossible by the observation made at the beginning of the proof. Therefore, we see that condition (I5) is satisfied, since $0 < \varrho_{n+1} < \varrho_n < \varrho_2 = \varrho_1 = 1$ for $n \ge 3$.

As for (I6), we note that if $\alpha, \beta \in]0,1[$ and $\alpha < \beta$, then for every $n \in \mathbf{N}^+$ we have that

$$(2n-1)! < \beta(2n-1)! + (1-\beta)(2n)! < \alpha(2n-1)! + (1-\alpha)(2n)! < (2n)!$$

and

$$\lim_{n \to \infty}((\alpha(2n-1)! + (1-\alpha)(2n)!) - (\beta(2n-1)! + (1-\beta)(2n)!)) = \infty,$$

which yields that $V_\alpha \subsetneq V_\beta \subseteq \mathbf{N}^+$. Hence, for all $n \in \mathbf{N}^+$ and $\varepsilon \in \mathbf{R}^+$ there exist $X, Y \subseteq \mathbf{H}$ such that $Y \subsetneq X$, $\iota(X) = n$, $\iota(Y) = n+1$, and $\varrho_{n+1} \le (1+\varepsilon)\mu^\star(Y)$: It is sufficient, by the claim we have proved above, to take $X := V_{1/n}$ and $Y = V_\alpha$ for some $\alpha \in \left[\max\left(\frac{1}{n+1}, \frac{1}{(1+\varepsilon)n}\right), \frac{1}{n}\right[$.

It follows that also condition (I6) is satisfied, and the proof of the lemma is thus complete. ∎

At long last, we have all what we need to confirm the independence of (F2).

Theorem 1. *Given $K \in [1, \infty]$, there exists a non-monotone, subadditive, (-1)-homogeneous, and translational invariant function $\theta^\star : \mathcal{P}(\mathbf{H}) \to \mathbf{R}$ with the property that $\sup_{X \in \mathcal{P}(\mathbf{H})} \theta^\star(X) = K$. In particular, there exists an upper quasi-density on $\mathbf{H}$ that is not an upper density.*

Proof. It follows at once from Lemmas 2 and 3 (we omit further details). ∎

4. EXAMPLES

We turn to examine a few examples that illustrate how to build uncountably many upper densities and, on the other hand, generalize some of the most important instances of the notion of "density" available in the literature.

EXAMPLE 4. Fix $\alpha \in [-1,\infty[$ and $a,b \in \mathbf{R}$ such that $\sum_{i\in F_n} |i|^\alpha \to \infty$ as $n\to\infty$, where we set $F_n := [an, bn] \cap \mathbf{H}$ for all n. We denote by $\mathfrak{F}$ the sequence $(F_n)_{n\ge 1}$ and consider the function

$$\mathsf{d}^\star(\mathfrak{F};\alpha) : \mathcal{P}(\mathbf{H}) \to \mathbf{R} : X \mapsto \limsup_{n\to\infty} \frac{\sum_{i\in X\cap F_n} |i|^\alpha}{\sum_{i\in F_n} |i|^\alpha},$$

with $\frac{0}{0} := 1$. It is seen (we omit details) that the dual of $\mathsf{d}^\star(\mathfrak{F};\alpha)$ is given by

$$\mathsf{d}_\star(\mathfrak{F};\alpha) : \mathcal{P}(\mathbf{H}) \to \mathbf{R} : X \mapsto \liminf_{n\to\infty} \frac{\sum_{i\in X\cap F_n} |i|^\alpha}{\sum_{i\in F_n} |i|^\alpha}.$$

Proposition 3. *The function $\mathsf{d}^\star(\mathfrak{F};\alpha)$ is an upper density.*

Proof. It is straightforward to check that $\mathsf{d}^\star(\mathfrak{F};\alpha)$ satisfies (F1), (F2), and (F3). As for (F6), fix $X \subseteq \mathbf{H}$ and $h,k \in \mathbf{N}^+$. Given $\varepsilon \in\,]0,1[$, there exists $n_\varepsilon \in \mathbf{N}^+$ such that $0 < (1-\varepsilon)|ik+h|^\alpha \le |ik|^\alpha \le (1+\varepsilon)|ik+h|^\alpha$ whenever $|i| \ge n_\varepsilon$. Together with the fact that $\mathsf{d}^\star(\mathfrak{F};\alpha)(S) = 0$ for every finite $S \subseteq \mathbf{H}$ (here we use that $\sum_{i\in F_n} |i|^\alpha = \infty$ as $n\to\infty$), this yields

$$\begin{cases} \mathsf{d}^\star(\mathfrak{F};\alpha)(k\cdot X + h) = \mathsf{d}^\star(\mathfrak{F};\alpha)(k\cdot X_\varepsilon + h) \\ (1-\varepsilon)k^\alpha u_\varepsilon \le \mathsf{d}^\star(\mathfrak{F};\alpha)(k\cdot X + h) \le (1+\varepsilon)k^\alpha u_\varepsilon \end{cases}, \tag{6}$$

where, for ease of notation, we have put

$$u_\varepsilon := \limsup_{n\to\infty} \frac{\sum_{i\in X_\varepsilon\cap(k^{-1}\cdot(F_n-h))} |i|^\alpha}{\sum_{i\in F_n} |i|^\alpha} = \limsup_{n\to\infty} \frac{\sum_{i\in X_\varepsilon\cap F_{\lfloor n/k\rfloor}} |i|^\alpha + \delta(n)}{\sum_{i\in F_n} |i|^\alpha}, \tag{7}$$

with $X_\varepsilon := \{i \in X : |i| \ge n_\varepsilon\}$ and $\delta(n) := \sum_{i\in X_\varepsilon\cap(k^{-1}\cdot(F_n-h))} |i|^\alpha - \sum_{i\in X_\varepsilon\cap F_{\lfloor n/k\rfloor}} |i|^\alpha$ for each n. Now, denote by $\Delta(n)$ the symmetric difference of $\mathbf{H}\cap(k^{-1}\cdot(F_n-h))$ and $F_{\lfloor n/k\rfloor}$. Then we have by the triangle inequality (we skip some details) that

$$0 \le \limsup_{n\to\infty} \frac{|\delta(n)|}{\sum_{i\in F_n} |i|^\alpha} \le \limsup_{n\to\infty} \frac{\sum_{i\in X_\varepsilon\cap\Delta(n)} |i|^\alpha}{\sum_{i\in F_n} |i|^\alpha} \le \limsup_{n\to\infty} \frac{\sum_{i\in\Delta(n)} |i|^\alpha}{\sum_{i\in F_n} |i|^\alpha} = 0,$$

which, combined with (7), implies

$$\begin{aligned} u_\varepsilon &= \limsup_{n\to\infty} \frac{\sum_{i\in X_\varepsilon\cap F_{\lfloor n/k\rfloor}} |i|^\alpha}{\sum_{i\in F_n} |i|^\alpha} = \frac{1}{k^{\alpha+1}} \limsup_{n\to\infty} \frac{\sum_{i\in X_\varepsilon\cap F_{\lfloor n/k\rfloor}} |i|^\alpha}{\sum_{i\in F_{\lfloor n/k\rfloor}} |i|^\alpha} \\ &= \frac{1}{k^{\alpha+1}} \limsup_{n\to\infty} \frac{\sum_{i\in X_\varepsilon\cap F_n} |i|^\alpha}{\sum_{i\in F_n} |i|^\alpha} = \frac{1}{k^{\alpha+1}}\mathsf{d}^\star(\mathfrak{F};\alpha)(X_\varepsilon) = \frac{1}{k^{\alpha+1}}\mathsf{d}^\star(\mathfrak{F};\alpha)(X), \end{aligned}$$

where we have used again that $\mathsf{d}^\star(\mathfrak{F};\alpha)(S) = 0$ for a finite $S \subseteq \mathbf{H}$ to argue that $\mathsf{d}^\star(\mathfrak{F};\alpha)(X_\varepsilon) = \mathsf{d}^\star(\mathfrak{F};\alpha)(X)$. So taking the limit of (6) as $\varepsilon \to 0^+$, we are led to the desired conclusion. ■

In continuity with [11, Definition 1.4], we call $\mathsf{d}^\star(\mathfrak{F};\alpha)$ and $\mathsf{d}_\star(\mathfrak{F};\alpha)$, respectively, the *upper* and *lower α-density relative to $\mathfrak{F}$*. In particular, if $a = 0$ and $b = 1$, then $\mathsf{d}^\star(\mathfrak{F};-1)$ is the upper logarithmic density, see [46, Chapter III.1, § 1.2], and $\mathsf{d}^\star(\mathfrak{F};0)$ is the upper asymptotic density.

To see that the construction we are considering is not vacuous (i.e., supplies upper densities that are different from the upper α-densities so far considered in the literature), fix a scaling coefficient $s > 1$ and let $(x_n)_{n\ge 1}$ a sequence of positive integers such that $x_n + 1 \le sx_n < x_{n+1}$ for every sufficiently large n and $x_n/x_{n+1} \to 0$ as $n \to \infty$ (e.g., we may assume $x_n := \lfloor s^n \rfloor!$ for all n). If $X := \bigcup_{n\ge 1} [\![x_n, sx_n]\!]$, then we have by Lemma 1 that $\mathsf{d}^\star(X) = 1/s$, while it is trivial that $\mathsf{d}^\star(\mathfrak{F}; 0)(X) = 1$ provided that $F_n := [\![n, sn]\!]$ for each n.

Another interesting example is offered by the upper Buck density (on $\mathbf{H}$), i.e., the function

$$\mathcal{P}(\mathbf{H}) \to \mathbf{R} : X \mapsto \inf_{S\in\mathscr{A}:X\subseteq S} \mathsf{d}^\star(S), \tag{8}$$

where $\mathscr{A}$ denotes, here and for the remainder of the section, the collection of all sets that can be expressed as a finite union of arithmetic progressions of $\mathbf{H}$, see, e.g., [4], [29, § 7] and [39] for the case $\mathbf{H} = \mathbf{N}^+$. In fact, we have the following generalization:

Example 5. Let $\mu^\star$ be an upper quasi-density on $\mathbf{H}$ and $\mathscr{C}$ a subfamily of $\mathcal{P}(\mathbf{H})$ such that:

- (B1) $\mathbf{H} \in \mathscr{C}$;
- (B2) $X \cup Y \in \mathscr{C}$ for all $X, Y \in \mathscr{C}$;
- (B3) $k \cdot X + h \in \mathscr{C}$, for some $X \subseteq \mathbf{H}$ and $h, k \in \mathbf{N}^+$, if and only if $X \in \mathscr{C}$;
- (B4) $X \cap (k \cdot \mathbf{H} + h) \in \mathscr{C}$ for all $X \in \mathscr{C}$ and $h, k \in \mathbf{N}^+$.

In particular, it is seen that $\mathscr{A}$ satisfies (B1)-(B4) if $\mathbf{H} = \mathbf{Z}$ or $\mathbf{H} = \mathbf{N}$, but not if $\mathbf{H} = \mathbf{N}^+$. On the other hand, it is not difficult to verify that conditions (B1)-(B4) are all satisfied by taking

$$\mathscr{C} = \mathscr{C}_\theta := \{X \cup Y : X \in \mathscr{A} \text{ and } \theta(Y) = 0\} \subseteq \mathcal{P}(\mathbf{H}), \tag{9}$$

provided θ is a function $\mathcal{P}(\mathbf{H}) \to \mathbf{R}$ such that:

- (i) $\theta(X) = 0$ for every finite $X \subseteq \mathbf{H}$;
- (ii) $\theta(X \cup Y) \le \theta(X) + \theta(Y)$ for all $X, Y \subseteq \mathbf{H}$ (so θ is nonnegative, cf. Proposition 2(vi));
- (iii) $\theta(X) \le \theta(Y)$ whenever $X \subseteq Y \subseteq \mathbf{H}$;
- (iv) $\theta(k \cdot X + h) = 0$, for some $X \subseteq \mathbf{H}$ and $h, k \in \mathbf{N}^+$, if and only if $\theta(X) = 0$.

E.g., these conditions are satisfied if θ is: an upper density on $\mathbf{H}$, see Proposition 6 below for (i); the characteristic function of the infinite subsets of $\mathbf{H}$, i.e., the function $\mathcal{P}(\mathbf{H}) \to \mathbf{R}$ mapping a set $X \subseteq \mathbf{H}$ to 0 if X is finite and to 0 otherwise (this is not an upper density), in which case $\mathscr{C}_\theta$ is the set of all subsets of $\mathcal{P}(\mathbf{H})$ that can be represented as a finite union of arithmetic progressions of $\mathbf{H}$, or differ from these by finitely many integers; the constant function $\mathcal{P}(\mathbf{H}) \to \mathbf{R} : X \mapsto 0$ (this is not an upper density either), in which case $\mathscr{C}_\theta = \mathcal{P}(\mathbf{H})$.

In fact, $\mathscr{C}$ is the basic ingredient for the definition of an upper density on $\mathbf{H}$ that is ultimately a generalization of the upper Buck density. To wit, we consider the function

$$\mathfrak{b}^\star(\mathscr{C}; \mu^\star) : \mathcal{P}(\mathbf{H}) \to \mathbf{R} : X \mapsto \inf_{S\in\mathscr{C}:X\subseteq S} \mu^\star(S),$$

which we denote by $\mathfrak{b}^\star$ whenever $\mathscr{C}$ and $\mu^\star$ are clear from the context, and is well defined by the fact that $\mathbf{H} \in \mathscr{C}$ and $\mu^\star(S) \in [0, 1]$ for all $S \in \mathscr{C}$ (by Remark 1).

Proposition 4. *The function $\mathfrak{b}^\star(\mathscr{C}; \mu^\star)$ is an upper density.*

Proof. First, it is clear that $\mathfrak{b}^\star(\mathbf{H}) = 1$, because $\mathbf{H} \subseteq S$ for some $S \in \mathscr{C}$ only if $S = \mathbf{H}$, and on the other hand, $\mathbf{H} \in \mathscr{C}$ by (B1) and $\mu^\star(\mathbf{H}) = 1$ by the fact that $\mu^\star$ satisfies (F1).

Second, if $X \subseteq Y \subseteq \mathbf{H}$ and $Y \subseteq S \in \mathscr{C}$, then of course $X \subseteq S \in \mathscr{C}$, with the result that

$$\mathfrak{b}^\star(X) = \inf_{S \in \mathscr{C}: X \subseteq S} \mu^\star(S) \le \inf_{S \in \mathscr{C}: Y \subseteq S} \mu^\star(S) = \mathfrak{b}^\star(Y).$$

Third, if $X, Y \subseteq \mathbf{H}$ and $S, T \in \mathscr{C}$ are such that $X \subseteq S$ and $Y \subseteq T$, then $S \cup T \in \mathscr{C}$ by (B2) and $X \cup Y \subseteq S \cup T$, which, together with the subadditivity of $\mu^\star$, gives that

$$\begin{aligned}\mathfrak{b}^\star(X \cup Y) &\le \inf_{S,T \in \mathscr{C}: X \subseteq S, Y \subseteq T} \mu^\star(S \cup T) \le \inf_{S,T \in \mathscr{C}: X \subseteq S, Y \subseteq T} (\mu^\star(S) + \mu^\star(T)) \\ &= \inf_{S \in \mathscr{C}: X \subseteq S} \mu^\star(S) + \inf_{T \in \mathscr{C}: Y \subseteq T} \mu^\star(T) = \mathfrak{b}^\star(X) + \mathfrak{b}^\star(Y).\end{aligned}$$

Lastly, pick $X \subseteq \mathbf{H}$ and $h, k \in \mathbf{N}^+$. If $X \subseteq S \in \mathscr{C}$, then (B3) yields $k \cdot X + h \subseteq k \cdot S + h \in \mathscr{C}$; conversely, if $k \cdot X + h \subseteq T$ for some $T \in \mathscr{C}$, then $k \cdot X + h \subseteq T \cap (k \cdot \mathbf{H} + h) = k \cdot S + h$ for some $S \in \mathbf{H}$, which implies, by (B3) and (B4), that $X \subseteq S \in \mathscr{C}$. Therefore, we find that

$$\mathfrak{b}^\star(k \cdot X + h) = \inf_{T \in \mathscr{C}: k \cdot X + h \subseteq T} \mu^\star(T) = \inf_{S \in \mathscr{C}: X \subseteq S} \mu^\star(k \cdot S + h) = \frac{1}{k} \mathfrak{b}^\star(X),$$

where we have used that $\mu^\star$ satisfies (F6). It follows that $\mathfrak{b}^\star$ is an upper density. ■

In view of Proposition 4, we will refer to $\mathfrak{b}^\star(\mathscr{C}; \mu^\star)$ as the *upper Buck density* (*on* $\mathbf{H}$) *relative to the pair* $(\mathscr{C}, \mu^\star)$. If $\mu^\star$ is the upper asymptotic density and $\mathscr{C}$ is the set $\mathscr{C}_\theta$ returned by (9) when θ is the characteristic function of the infinite subsets of $\mathbf{H}$, then $\mathfrak{b}^\star(\mathscr{C}; \mu^\star)$ is the upper Buck density, as given by (8); in particular, Proposition 4 generalizes [31, Corollaries 2 and 3].

It is perhaps interesting to note that the definition of $\mathfrak{b}^\star(\mathscr{C}; \mu^\star)$ produces a "smoothing effect" on $\mu^\star$, inasmuch as the former is monotone no matter if so is the latter.

In addition, we have the following result, which, together with Proposition 14 in § 7, proves that the (common) definition of the upper Buck density on $\mathbf{N}^+$ can be (slightly) simplified by establishing, as we do, that the *upper Buck density on* $\mathbf{H}$ is given by $\mathfrak{b}^\star(\mathscr{A}; \mathsf{d}^\star)$ if $\mathbf{H} = \mathbf{Z}$, and by the restriction to $\mathcal{P}(\mathbf{H})$ of the upper Buck density on $\mathbf{Z}$ otherwise (recall that $\mathscr{A}$ does not satisfy (B1)-(B4) if $\mathbf{H} = \mathbf{N}^+$, and $\mathsf{d}^\star$ accounts only for the positive part of a subset of $\mathbf{H}$).

Accordingly, we take the Buck density on $\mathbf{H}$ to be the density induced by $\mathfrak{b}^\star(\mathscr{A}; \mathsf{d}^\star)$, and refer to the dual of $\mathfrak{b}^\star(\mathscr{A}; \mathsf{d}^\star)$ as the lower Buck density on $\mathbf{H}$.

Proposition 5. *Let* $\mathscr{C}^\sharp := \{X \cup \mathcal{H} : X \in \mathscr{C}, \mathcal{H} \subseteq \mathbf{H}, |\mathcal{H}| < \infty\}$. *Then* $\mathfrak{b}^\star(\mathscr{C}; \mu^\star) = \mathfrak{b}^\star(\mathscr{C}^\sharp; \mu^\star)$.

Proof. Pick $X \subseteq \mathbf{H}$. Since $\mathscr{C} \subseteq \mathscr{C}^\sharp$, it is immediate that $\mathfrak{b}^\star(\mathscr{C}^\sharp; \mu^\star)(X) \le \mathfrak{b}^\star(\mathscr{C}; \mu^\star)(X)$, where we use, as in the proof of Proposition 4, that if $\varnothing \ne A \subseteq B \subseteq \mathbf{R}$ then $\inf(B) \le \inf(A)$. We are left to prove that $\mathfrak{b}^\star(\mathscr{C}; \mu^\star)(X) \le \mathfrak{b}^\star(\mathscr{C}^\sharp; \mu^\star)(X)$.

For this, fix a real $\varepsilon > 0$. By definition of the infimum, there exists $T \in \mathscr{C}^\sharp$ for which $X \subseteq T$ and $\mu^\star(T) \le \mathfrak{b}^\star(\mathscr{C}^\sharp; \mu^\star)(X) + \frac{\varepsilon}{2}$. On the other hand, $T \in \mathscr{C}^\sharp$ if and only if $T = Y \cup \mathcal{H}$ for some $Y \in \mathscr{C}$ and $\mathcal{H} \subseteq \mathbf{H}$ with $|\mathcal{H}| < \infty$. Set $\mathcal{V} := \bigcup_{h \in \mathcal{H}} (k \cdot \mathbf{H} + h)$, where k is a positive integer with $\frac{1}{k}|\mathcal{H}| \le \frac{\varepsilon}{2}$, and notice that $(Y + k) \cup \mathcal{V} = (T + k) \cup \mathcal{V}$.

It then follows from the above and conditions (B1)-(B3) that $X + k \subseteq (T + k) \cup \mathcal{V} \in \mathscr{C}$, and this in turn implies, by the fact that $\mathfrak{b}^\star(\mathscr{C}; \mu^\star)(X)$ is translational invariant (by Proposition 4)

and $\mu^\star$ is translational invariant and subadditive (by hypothesis), that

$$\mathfrak{b}^\star(\mathscr{C};\mu^\star)(X) = \mathfrak{b}^\star(\mathscr{C};\mu^\star)(X+k) \le \mu^\star(T+k) + \mu^\star(\mathcal{V}) = \mu^\star(T) + \frac{|\mathcal{H}|}{k} \le \mathfrak{b}^\star(\mathscr{C}^\sharp;\mu^\star)(X) + \varepsilon,$$

which, by the arbitrariness of ε, is enough to complete the proof. ■

We are left with the question of providing a "convenient expression" for the lower dual of $\mathfrak{b}^\star(\mathscr{C};\mu^\star)$, here denoted by $\mathfrak{b}_\star(\mathscr{C};\mu^\star)$ and called the lower Buck density relative to pair $(\mathscr{C},\mu^\star)$. This does not seem to be feasible in general, but if $\mathscr{C}$ is closed under complementation, viz.

(B5) $X^c \in \mathscr{C}$ whenever $X \in \mathscr{C}$,

then it is not difficult to verify that $\mathfrak{b}_\star(\mathscr{C};\mu^\star)(X)$ is given by the function

$$\mathcal{P}(\mathbf{H}) \to \mathbf{R} : X \mapsto \sup_{T\in\mathscr{C}:T\subseteq X} \mu_\star(T),$$

with $\mu_\star$ being the lower dual of $\mu^\star$. In fact, if $\mathscr{C}$ satisfies (B5) then we have, for all $X \subseteq \mathbf{H}$, that

$$\begin{aligned}\mathfrak{b}_\star(\mathscr{C};\mu^\star)(X) &:= 1 - \mathfrak{b}^\star(\mathscr{C};\mu^\star)(X^c) = 1 - \inf_{S\in\mathscr{C}:X^c\subseteq S} \mu^\star(S) = 1 - \inf_{S\in\mathscr{C}:S^c\subseteq X} \mu^\star(S) \\ &= 1 - \inf_{T\in\mathscr{C}:T\subseteq X} \mu^\star(T^c) = \sup_{T\in\mathscr{C}:T\subseteq X} (1-\mu^\star(T^c)) = \sup_{T\in\mathscr{C}:T\subseteq X} \mu_\star(T).\end{aligned}$$

Lastly, we note that the above construction is not vacuous, in the sense that, for some choice of $\mu^\star$ and $\mathscr{C}$, we have $\mu^\star = \mathfrak{b}^\star(\mathcal{P}(\mathbf{H});\mu^\star) \ne \mathfrak{b}^\star(\mathscr{C};\mu^\star) \ne \mathfrak{b}^\star(\mathscr{A};\mu^\star)$. Indeed, let $\mathsf{ld}^\star$ be the upper logarithmic density on $\mathbf{H}$ (see Example 4). It follows by Proposition 11 (we omit details) that there exists $X \subseteq \mathbf{N}^+$ such that $\mathsf{ld}^\star(X) = \mathsf{d}^\star(X) = 0$, but $\mathfrak{b}^\star(\mathscr{A};\mathsf{d}^\star)(X) = 1$. On the other hand, we get from the comments at the beginning of § 7 that there is $Y \subseteq \mathbf{N}^+$ such that $\mathsf{ld}^\star(Y) = 0$ and $\mathsf{d}^\star(Y) = 1$. Hence, we find that $\mathsf{d}^\star \ne \mathfrak{b}^\star(\mathscr{C};\mathsf{d}^\star) \ne \mathfrak{b}^\star(\mathscr{A};\mathsf{d}^\star)$, where, consistently with (9), we set $\mathscr{C} := \{S \cup T : S \in \mathscr{A} \text{ and } \mathsf{ld}^\star(T) = 0\}$.

Our last example is another classic of the "literature on densities".

EXAMPLE 6. Denote by ζ the function $]1,\infty[\to \mathbf{R} : s \mapsto \sum_{n\ge1} n^{-s}$, namely, the restriction of the Riemann zeta function to the interval $]1,\infty[$. Then consider the function

$$\mathfrak{a}^\star : \mathcal{P}(\mathbf{H}) \to \mathbf{R} : X \mapsto \limsup_{s\to1^+} \frac{1}{\zeta(s)} \sum_{i\in X^+} \frac{1}{i^s}.$$

We claim that $\mathfrak{a}^\star$ is an upper density, which we refer to as the *upper analytic density* (on $\mathbf{H}$) for consistency with [46, Part III, § 1.3], where the focus is on the case $\mathbf{H} = \mathbf{N}^+$.

In fact, it is straightforward to check that $\mathfrak{a}^\star$ satisfies (F1)-(F4). As for (F5), fix $X \subseteq \mathbf{H}$ and $h \in \mathbf{N}$, and pick $\varepsilon \in]0,1[$. There exists $n_\varepsilon \in \mathbf{N}^+$ such that

$$0 < (1-\varepsilon)|i+h| \le |i| \le (1+\varepsilon)|i+h| \tag{10}$$

for $|i| \ge n_\varepsilon$. Set $X_\varepsilon := \{i \in X : i \ge n_\varepsilon\}$. Then $\mathfrak{a}^\star(S) = 0$, and hence $\mathfrak{a}^\star(T) = \mathfrak{a}^\star(S \cup T)$, for all $S, T \subseteq \mathbf{H}$ with $|S| < \infty$. Thus $\mathfrak{a}^\star(X) = \mathfrak{a}^\star(X_\varepsilon)$, and by (10) we have

$$\limsup_{s\to1^+} \frac{1}{(1+\varepsilon)^s\zeta(s)} \sum_{i\in X_\varepsilon} \frac{1}{(i+h)^s} \le \mathfrak{a}^\star(X) \le \limsup_{s\to1^+} \frac{1}{(1-\varepsilon)^s\zeta(s)} \sum_{i\in X_\varepsilon} \frac{1}{(i+h)^s}.$$

This, in the limit as $\varepsilon \to 0^+$, yields that $\mathfrak{a}^\star(X) = \mathfrak{a}^\star(X_\varepsilon + h) = \mathfrak{a}^\star(X+h)$, where we have used again that $\mathfrak{a}^\star$ is "invariant under union with finite sets".

Therefore $\mathfrak{a}^\star$ is an upper density, whose lower dual (we omit details) is given by

$$\mathfrak{a}_\star : \mathcal{P}(\mathbf{H}) \to \mathbf{R} : X \mapsto \liminf_{s\to 1^+} \frac{1}{\zeta(s)} \sum_{i\in X^+} \frac{1}{i^s}.$$

Later on, we will see how to construct new upper densities from old ones, see the discussion at the beginning of § 6 and Proposition 10 below for details.

5. Range of upper and lower densities

There are a number of natural questions that we may want to ask about upper densities. In the light of Remark 1, one of the most basic of them is probably the following:

Question 1. Let $\mu^\star$ be an upper density on $\mathbf{H}$. Is it true that $\mathrm{Im}(\mu^\star) = [0,1]$?

We will see that the answer to Question 1 is affirmative, see Theorem 2. In fact, we will show that the image of every quasi-density on $\mathbf{H}$ is the whole interval $[0,1]$. As a consequence, the image of every upper and lower quasi-density is the interval $[0,1]$.

This generalizes [4, Theorem 6] and [31, Theorem 5] (the case of the upper Buck density on $\mathbf{N}^+$), as well as similar conclusions that are known to hold for other classical densities (cf. also Question 3 and [25, § 3]), and is kind of an analog for upper quasi-densities of a theorem of A. A. Liapounoff [24, Theorem 1] on the convexity of the range of a non-atomic countably additive vector measure (with values in $\mathbf{R}^n$).

Proposition 6. *Let $\mu^\star$ be a function $\mathcal{P}(\mathbf{H}) \to \mathbf{R}$ that satisfies axioms (F1), (F3) and (F6). If X is a finite subset of $\mathbf{H}$, then $\mu^\star(X) = 0$.*

Proof. We know from Proposition 2(iii) that $\mu^\star(\varnothing)$ is zero, and the same is true for $\mu^\star(\{0\})$ in the case $0 \in \mathbf{H}$. In addition, it follows from (F6) that, for every $k \ge 1$,

$$\mu^\star(\{1\}) = \mu^\star(\{1\} + k - 1) = \mu^\star(\{k\}) = \mu^\star(k \cdot \{1\}) = \frac{1}{k}\mu^\star(\{1\}),$$

which is possible only if $\mu^\star(\{1\}) = 0$. Thus, $\mu^\star(\{k\}) = 0$ for all $k \ge 0$, and on the other hand, if $\mathbf{H} = \mathbf{Z}$ and $k \le 0$ then $\mu^\star(\{k\}) = \mu^\star(\{k\} + (-k)) = \mu^\star(\{0\}) = 0$.

With this in hand, let X be a finite subset of $\mathbf{H}$. Then, we get by (F3), Proposition 2(vi) and the considerations above that $0 \le \mu^\star(X) \le \sum_{x\in X} \mu^\star(\{x\}) = 0$, which completes the proof. ∎

Incidentally, observe that, in view of Example 3, conditions (F1)-(F4) alone are not sufficient for Proposition 6 to hold, and on the other hand, the proposition is independent of (F2).

Proposition 7. *Let $\mu^\star$ be given as in Proposition 6, and for a fixed $k \in \mathbf{N}^+$ let $h_1, \ldots, h_n \in \mathbf{N}$ be such that $h_i \not\equiv h_j \bmod k$ for $1 \le i < j \le n$ and set $X := \bigcup_{i=1}^n (k \cdot \mathbf{H} + h_i)$. Then, for every finite $\mathcal{V} \subseteq \mathbf{H}$ we have $\mu^\star(X \cup \mathcal{V}) = \mu^\star(X \setminus \mathcal{V}) = \mu_\star(X \cup \mathcal{V}) = \mu_\star(X \setminus \mathcal{V}) = \frac{n}{k}$, where $\mu_\star$ is the lower dual of $\mu^\star$.*

Proof. Let l_i be, for each $i = 1, \ldots, n$, the remainder of the integer division of h_i by k (in such a way that $0 \le l_i < k$), and set $Y := \bigcup_{l\in\mathcal{H}} (k \cdot \mathbf{H} + l)$, where $\mathcal{H} := [\![0, k-1]\!] \setminus \{l_1, \ldots, l_n\}$.

Clearly, $\mathbf{H} = X \cup Y \cup S$ for some finite $S \subseteq \mathbf{H}$. Therefore, we have from axioms (F1), (F3), and (F6) and Propositions 6 and 2(ii) that, however we choose a finite $\mathcal{V} \subseteq \mathbf{H}$,

$$1 = \mu^\star(\mathbf{H}) \le \mu^\star(X \cup \mathcal{V}) + \mu^\star(Y \cup S) \le \mu^\star(X) + \mu^\star(Y) + \mu^\star(S) + \mu^\star(\mathcal{V})$$
$$= \mu^\star(X) + \mu^\star(Y) \le n\mu^\star(k \cdot \mathbf{H}) + (k-n)\mu^\star(k \cdot \mathbf{H}) = k\mu^\star(k \cdot \mathbf{H}) = 1,$$

which is possible only if $\mu^\star(X \cup \mathcal{V}) = \mu^\star(X) = n\mu^\star(k \cdot \mathbf{H}) = \frac{n}{k}$.

On the other hand, for every finite $\mathcal{V} \subseteq \mathbf{H}$ there exist $h \in \mathbf{N}$ and a finite $\mathcal{W} \subseteq \mathbf{H}$ such that $X \setminus \mathcal{V} = (X + h) \cup \mathcal{W}$, so we get by (F5) and the above that $\mu^\star(X \setminus \mathcal{V}) = \mu^\star((X \cup h) \cup \mathcal{W}) = \frac{n}{k}$, and hence we are done with the part of the claim relative to $\mu^\star$.

As for $\mu_\star$, it is straightforward that if $\mathcal{V}$ is a finite subset of $\mathbf{H}$ then $|Y \triangle (X \cup \mathcal{V})^c| < \infty$ and $|Y \triangle (X \setminus \mathcal{V})^c| < \infty$, which, together with the first part, yields $\mu^\star((X \cup \mathcal{V})^c) = \mu^\star((X \setminus \mathcal{V})^c) = 1 - \frac{n}{k}$, and consequently $\mu_\star(X \cup \mathcal{V}) = \mu_\star(X \setminus \mathcal{V}) = \frac{n}{k}$. ■

In fact, Proposition 7 can be thought of as a supplement to Proposition 1(ii) (in view of Proposition 6), and is already enough to imply the following corollary (we omit further details), which falls short of an answer to Question 1, but will be useful later in the proof of Theorem 2.

Corollary 1. *Let μ be an upper quasi-density on $\mathbf{H}$. Then $\mathbf{Q} \cap [0, 1] \subseteq \mathrm{Im}(\mu)$.*

The next result is essentially an extension of Proposition 7.

Proposition 8. *Let $\mu^\star$ be as in Proposition 6, and assume that $(k_i)_{i\ge1}$ and $(h_i)_{i\ge1}$ are integer sequences with the property that:*

(i) *$k_i \ge 1$ and $k_i \mid k_{i+1}$ for each $i \in \mathbf{N}^+$;*
(ii) *Given $i, j \in \mathbf{N}^+$ with $i < j$, there exists no $x \in \mathbf{Z}$ such that $k_i x + h_i \equiv h_j \bmod k_j$.*

Then $\mu^\star(X_n) = \sum_{i=1}^n \frac{1}{k_i}$, where we set $X_n := \bigcup_{i=1}^n (k_i \cdot \mathbf{H} + h_i)$.

Proof. Fix $n \in \mathbf{N}^+$. By condition (i), we can write, for each $i = 1, \dots, n$, that

$$k_i \cdot \mathbf{H} + h_i = \bigcup_{l=0}^{k_i^{-1}k_n - 1} (k_n \cdot \mathbf{H} + k_i l + h_i),$$

with the result that

$$X_n = \bigcup_{i=1}^{n} \bigcup_{l=0}^{k_i^{-1}k_n - 1} (k_n \cdot \mathbf{H} + k_i l + h_i). \tag{11}$$

On the other hand, if $1 \le i < j \le n$ then $k_i l_i + h_i \not\equiv k_j l_j + h_j \bmod k_n$ for all $l_i \in [\![0, k_i^{-1}k_n - 1]\!]$ and $l_j \in [\![0, k_j^{-1}k_n - 1]\!]$, as otherwise we would have that l_i is an integer solution to the congruence $k_i x + h_i \equiv h_j \bmod k_j$ (by the fact that $k_j \mid k_n$), contradicting condition (ii).

Thus, it follows from (11) and Proposition 7 that $\mu^\star(X_n) = \frac{1}{k_n}\sum_{i=1}^n k_i^{-1}k_n = \sum_{i=1}^n \frac{1}{k_i}$. ■

While (F2) is, by Theorem 1, independent of (F1), (F3) and (F6), the latter conditions are almost sufficient to prove a weak form of (F2), as we do in the next two statements, where $\mathscr{A}^\sharp$ stands for the set of all subsets X of $\mathbf{H}$ that are finite unions of arithmetic progressions of $\mathbf{H}$, or differ from these by a finite number of integers (in particular, $\varnothing \in \mathscr{A}^\sharp$), cf. Example 5.

Proposition 9. *Let $\mu^\star$ be an upper quasi-density on $\mathbf{H}$, and pick $X \in \mathcal{P}(\mathbf{H})$ and $Y \in \mathscr{A}^\sharp$ such that $X \subseteq Y$. Then $\mu^\star(X) \le \mu^\star(Y)$.*

Proof. Since $Y \in \mathscr{A}^\sharp$, there exist $k \in \mathbf{N}^+$ and $\mathcal{H} \subseteq [\![0, k-1]\!]$ such that the symmetric difference of Y and $\bigcup_{h \in \mathcal{H}}(k \cdot \mathbf{H} + h)$ is finite. Using that $X \subseteq Y$, this yields that the relative complement of $\bigcup_{h \in \mathcal{H}} X_h$ in X, where $X_h := X \cap (k \cdot \mathbf{H} + h) \subseteq X$ for each $h \in \mathcal{H}$, is finite too. Therefore, we get $\mu^\star(X) \le \sum_{h \in \mathcal{H}} \mu^\star(X_h)$ by Propositions 6 and 2(ii), and $\mu^\star(Y) = \frac{1}{k}|\mathcal{H}|$ by Proposition 7.

On the other hand, we have that, however we choose $h \in \mathcal{H}$, there is a set $S_h \subseteq \mathbf{H}$ for which $X_h = k \cdot S_h + h$, so we get from the above, (F6) and the fact that $\mathrm{Im}(\mu^\star) \subseteq [0,1]$ that

$$\mu^\star(X) \le \sum_{h \in \mathcal{H}} \mu^\star(X_h) = \sum_{h \in \mathcal{H}} \mu^\star(k \cdot S_h + h) = \frac{1}{k} \sum_{h \in \mathcal{H}} \mu^\star(S_h) \le \mu^\star(Y). \qquad \blacksquare$$

Corollary 2. *Pick $X \subseteq \mathbf{H}$ and $Y, Z \in \mathscr{A}^\sharp$ such that $Y \subseteq X \subseteq Z$, and let μ be the quasi-density induced by an upper quasi-density $\mu^\star$ on $\mathbf{H}$ and $\mu_\star$ the lower dual of $\mu^\star$. Then $Y, Z \in \mathrm{dom}(\mu)$ and $\mu(Y) \le \mu_\star(X) \le \mu^\star(X) \le \mu(Z)$.*

Proof. First, $Y \subseteq X$ implies $X^c \subseteq Y^c$, and it is clear that $Y^c \in \mathscr{A}^\sharp$. So we get from Propositions 2(vi) and 9 that $\mu_\star(X) \le \mu^\star(X) \le \mu^\star(Z)$ and $\mu^\star(X^c) \le \mu^\star(Y^c)$, and the latter inequality gives $\mu_\star(Y) \le \mu_\star(X)$. This is enough to complete the proof, since we know from Proposition 7 that $Y, Z \in \mathrm{dom}(\mu)$, and therefore $\mu_\star(Y) = \mu(Y)$ and $\mu^\star(Z) = \mu(Z)$. $\blacksquare$

At long last, we are ready to answer Question 1.

Theorem 2. *Let μ be the quasi-density induced by an upper quasi-density $\mu^\star$ on $\mathbf{H}$. Then the range of μ is $[0,1]$. In particular, $\mathrm{Im}(\mu^\star) = \mathrm{Im}(\mu_\star) = [0,1]$, where $\mu_\star$ is the lower dual of $\mu^\star$.*

Proof. Remark 1 and Corollary 1 yield $\mathbf{Q} \cap [0,1] \subseteq \mathrm{Im}(\mu) \subseteq [0,1]$. So, fix an irrational number $\alpha \in [0,1]$. Then, there is determined an increasing sequence $(a_i)_{i \ge 1}$ of positive integers such that $\alpha = \sum_{i \ge 1} 2^{-a_i}$. Accordingly, let X_i denote, for each $i \in \mathbf{N}^+$, the set $X_i := 2^{a_i} \cdot \mathbf{H} + r_i$, where, for ease of notation, we put $r_i := \sum_{j=1}^{i-1} 2^{a_j - 1}$. Let $X := \bigcup_{i \ge 1} X_i$.

Given $n \in \mathbf{N}^+$, we note that $X_i \subseteq 2^{a_n - 1} \cdot \mathbf{H} + r_n$ for every $i \ge n$, because $x \in X_i$ if and only there exists $y \in \mathbf{H}$ such that $x = 2^{a_i} y + \sum_{j=1}^{i-1} 2^{a_j - 1}$, from which it is found that $x = 2^{a_n - 1} z + r_n$ for some $z \in \mathbf{H}$. Taking $Y_n := \bigcup_{i=1}^n X_i$, it follows that

$$Y_n \subseteq X \subseteq Y_n \cup (2^{a_n - 1} \cdot \mathbf{H} + r_n),$$

which in turn implies by Corollary 2 and axiom (F3) that

$$\mu(Y_n) \le \mu_\star(X) \le \mu^\star(X) \le \mu(Y_n \cup (2^{a_n - 1} \cdot \mathbf{H} + r_n)) \le \mu(Y_n) + \mu(2^{a_n - 1} \cdot \mathbf{H} + r_n). \qquad (12)$$

On the other hand, it is seen that, however we choose $i, j \in \mathbf{N}^+$ with $i < j$, there does not exist any $x \in \mathbf{Z}$ such that $2^{a_i} x + r_i \equiv r_j \bmod 2^{a_j}$, as otherwise we would have

$$2^{a_i} x \equiv \sum_{l=i}^{j-1} 2^{a_l - 1} \bmod 2^{a_j},$$

viz. $2x \equiv \sum_{l=i}^{j-1} 2^{a_l - a_i} \bmod 2^{a_j - a_i + 1}$, which is impossible, because $\sum_{l=i}^{j-1} 2^{a_l - a_i}$ is an odd integer and $a_j - a_i + 1 > 0$. It follows from Proposition 8, equation (12) and axiom (F6) that

$$\sum_{i=1}^n \frac{1}{2^{a_i}} = \mu(Y_n) \le \mu_\star(X) \le \mu^\star(X) \le \frac{1}{2^{a_n - 1}} + \sum_{i=1}^n \frac{1}{2^{a_i}},$$

so passing to the limit as $n \to \infty$, we get by the squeeze theorem that $\mu_\star(X) = \mu^\star(X) = \alpha$. Thus, $X \in \mathrm{dom}(\mu)$ and $\mu(X) = \alpha$, which completes the proof by the arbitrariness of α. ∎

Incidentally, it has been recently shown by the authors in [21, Theorem 1] that upper and lower quasi-densities have a kind of intermediate value property, which is actually much stronger than the "In particular" part of Theorem 2 (cf. also Question 4 below).

6. Some structural results

Let $\mathscr{G}$ be a subset of $\mathscr{F} := \{(\text{F1}), \ldots, (\text{F5})\}$, where axioms (F1), …, (F5) are viewed as words of a suitable formal language; in particular, we write $\mathscr{F}_1$ for $\mathscr{F} \setminus \{(\text{F1})\}$, $\mathscr{F}_2$ for $\mathscr{F} \setminus \{(\text{F2})\}$, and so forth. We denote by $\mathcal{M}^\star(\mathscr{G})$ the set of all functions $\mu^\star : \mathcal{P}(\mathbf{H}) \to \mathbf{R}$ that satisfy each of the axioms in $\mathscr{G}$ and the condition $\mathrm{Im}(\mu^\star) \subseteq [0,1]$, and by $\mathcal{M}_\star(\mathscr{G})$ the set of the duals of the functions in $\mathcal{M}^\star(\mathscr{G})$. In particular, $\mathcal{M}^\star(\mathscr{F})$ and $\mathcal{M}^\star(\mathscr{F}_2)$ are, respectively, the sets of all upper densities and upper quasi-densities (on $\mathbf{H}$). We may ask the following (vague) question:

Question 2. Is there anything interesting about the "structure" of $\mathcal{M}^\star(\mathscr{G})$ and $\mathcal{M}_\star(\mathscr{G})$?

For a partial answer, we regard $\mathcal{M}^\star(\mathscr{G})$ and $\mathcal{M}_\star(\mathscr{G})$ as subsets of $\mathcal{B}(\mathcal{P}(\mathbf{H}), \mathbf{R})$, the real vector space of all bounded functions $f : \mathcal{P}(\mathbf{H}) \to \mathbf{R}$, endowed with the (partial) order $\preceq$ given by $f \preceq g$ if and only if $f(X) \le g(X)$ for all $X \subseteq \mathbf{H}$; we write $f \prec g$ if $f \preceq g$ and $f \ne g$.

Unless noted otherwise, any linear property of $\mathcal{M}^\star(\mathscr{G})$ and $\mathcal{M}_\star(\mathscr{G})$ will be referred to the space $\mathcal{B}(\mathcal{P}(\mathbf{H}), \mathbf{R})$, any order-theoretic property to the order $\preceq$, and any topological property to the topology of pointwise convergence. In particular, a subset $\mathcal{F}$ of nonnegative and uniformly bounded functions of $\mathcal{B}(\mathcal{P}(\mathbf{H}), \mathbf{R})$ is said to be countably q-convex for some $q \in \mathbf{R}^+$ if, however we choose a $[0,1]$-valued sequence $(\alpha_n)_{n\ge1}$ such that $\sum_{n=1}^\infty \alpha_n = 1$ and an $\mathcal{F}$-valued sequence $(f_n)_{n\ge1}$, it holds that the function

$$f : \mathcal{P}(\mathbf{H}) \to \mathbf{R} : X \mapsto \left(\sum_{n=1}^\infty \alpha_n (f_n(X))^q\right)^{1/q}$$

still belongs to $\mathcal{F}$. Note that f is well defined. In particular, "$\mathcal{F}$ being sigma-convex" is the same as "$\mathcal{F}$ being countably 1-convex" (which in turn implies that $\mathcal{F}$ is convex).

Proposition 10. *Let $q \in [1, \infty[$. Then $\mathcal{M}^\star(\mathscr{G})$ is a countably q-convex set.*

Proof. Let $\mu^\star := \sum_{n=1}^\infty \alpha_n \mu_n^q$, where $(\alpha_n)_{n\ge1}$ is a $[0,1]$-valued sequence with $\sum_{n=1}^\infty \alpha_n = 1$ and $(\mu_n)_{n\ge1}$ is an $\mathcal{M}^\star(\mathscr{G})$-valued sequence. We leave as an exercise for the reader to check that $\mu^\star \in \mathcal{M}^\star(\mathscr{G})$. We just note that, if $X, Y \subseteq \mathbf{H}$ and $\mu_n(X \cup Y) \le \mu_n(X) + \mu_n(Y)$ for each n, then the subadditivity of $\mu^\star$ is a consequence of Minkowski's inequality for sums. ∎

The above proposition implies, along with Example 4, that $\mathcal{M}^\star(\mathscr{F})$ has at least the same cardinality of $\mathbf{R}$. Moreover, it leads to the following corollary, of which we omit the proof:

Corollary 3. *Both $\mathcal{M}^\star(\mathscr{G})$ and $\mathcal{M}_\star(\mathscr{G})$ are sigma-convex sets.*

On the other hand, we have from Propositions 1(iv), 2, 6, and 14 that if $\mu^\star$ is an upper density with lower dual $\mu_\star$, then $\mu^\star$ and $\mu_\star$ satisfy axioms (L2)-(L9) with $\mathbf{H} = \mathbf{N}^+$, $\delta_\star = \mu_\star$,

and $\delta^\star = \mu^\star$ (in the notation of the introduction). Thus, it is natural to ask whether a lower density in the sense of this paper needs to be also a lower density in the sense of Freedman and Sember's work [8]. The next example shows that this is not the case, and we have already noted in Example 1 that the converse does not hold either.

Example 7. Let f and g be two upper densities, and let $\alpha \in [0,1]$ and $q \in [1,\infty[$. We have from Proposition 10 that the function $h^\star := (\alpha f^q + (1-\alpha)g^q)^{\frac{1}{q}}$ is an upper density too.

In particular, assume from now on that f is the upper asymptotic density and g the upper Banach density. Next, fix $a \in]0,1]$, and set $V_a := \bigcup_{n\ge 1} [\![a(2n-1)! + (1-a)(2n)!, (2n)!]\!]$ and

$$X := V_a \cup (V_a^c \cap (2\cdot \mathbf{H})) \quad \text{and} \quad Y := V_a \cup (V_a^c \cap (2\cdot\mathbf{H}+1)) \cup \{1\}.$$

It is clear that $X \cup Y = \mathbf{H}$ and $X \cap Y = V_a$. Moreover, we get from Lemma 1 that

$$f(X) \le f(V_a) + f(V_a^c \cap (2\cdot\mathbf{H})) \le a + \frac{1}{2},$$

and similarly $f(Y) \le a + \frac{1}{2}$ and $f(V_a) = a$. On the other hand, V_a contains arbitrarily large intervals of consecutive integers, hence $g(X) = g(Y) = g(V_a) = 1$. It follows that

$$\begin{cases} 1 + h^\star(X\cap Y) = 1 + (\alpha + a^q(1-\alpha))^{\frac{1}{q}} \\ h^\star(X) + h^\star(Y) \le 2\cdot(\alpha + (a + 1/2)^q(1-\alpha))^{\frac{1}{q}} \end{cases}.$$

With this in hand, suppose by contradiction that $1+h^\star(A\cup B) \le h^\star(A)+h^\star(B)$ for all $A,B\subseteq\mathbf{H}$ such that $A\cup B = \mathbf{H}$ (regardless of the actual values of the parameters a, α and q), which is equivalent to saying that the conjugate of $h^\star$ satisfies (L1). Then, we have from the above that

$$1 + \alpha^{\frac{1}{q}} \le 2\cdot(\alpha + (a+1/2)^q(1-\alpha))^{\frac{1}{q}},$$

which ultimately implies, in the limit as $a \to 0^+$, that $1+\alpha^{\frac{1}{q}} \le (2^q\alpha + 1 - \alpha)^{\frac{1}{q}}$ for all $\alpha \in [0,1]$ and $q \in [1,\infty[$. This, however, is false $\big($e.g., let $\alpha = \frac{1}{2}$ and $q = 2\big)$.

Now we show that there exists $\mu^\star \in \mathcal{M}^\star(\mathscr{F})$ that is both a maximum and an extremal element of $\mathcal{M}^\star(\mathscr{F}_2)$, where "maximum" means that $\theta^\star \preceq \mu^\star$ for all $\theta^\star \in \mathcal{M}^\star(\mathscr{F}_2)$, and "extremal" means that there are no $\theta_1,\theta_2 \in \mathcal{M}^\star(\mathscr{F}_2)$ such that $\theta_1 \ne \theta_2$ and $\mu^\star = \alpha\theta_1 + (1-\alpha)\theta_2$ for some $\alpha \in]0,1[$.

But first we need the following proposition, which generalizes and extends a criterion used in [4, § 3, p. 563] to prove that the upper Buck density of the set of perfect squares is zero.

Proposition 11. *Fix $X \subseteq \mathbf{H}$, let $(\mu_\star,\mu^\star)$ be a conjugate pair on $\mathbf{H}$ such that $\mu^\star$ is an upper quasi-density, and for $k \ge 1$ and $S \subseteq \mathbf{H}$ denote by $w_k(S)$ the number of residues $h \in [\![0,k-1]\!]$ with the property that $\mu^\star(S\cap(k\cdot\mathbf{H}+h)) > 0$. Then, for all $k \in \mathbf{N}^+$ we have*

$$1 - \frac{w_k(X^c)}{k} \le \mu_\star(X) \le \mu^\star(X) \le \frac{w_k(X)}{k}. \tag{13}$$

Proof. Pick $k \in \mathbf{N}^+$ and $S \subseteq \mathbf{H}$, and let $\mathcal{W}_k(S)$ be the set of all integers $h \in [\![0,k-1]\!]$ for which $\mu^\star(S\cap(k\cdot\mathbf{H}+h)) > 0$, so that $w_k(S) = |\mathcal{W}_k(S)|$.

However we choose $h \in [\![0,k-1]\!]$, there exists $S_h \subseteq \mathbf{H}$ such that $S\cap(k\cdot\mathbf{H}+h) = k\cdot S_h + h$. Therefore, Propositions 2(ii) and 6 and axioms (F3) and (F6) imply that

$$\mu^\star(S) \le \sum_{h=0}^{k-1}\mu^\star(S\cap(k\cdot\mathbf{H}+h)) = \sum_{h\in\mathcal{W}_k(S)}\mu^\star(k\cdot S_h+h) = \sum_{h\in\mathcal{W}_k(S)}\frac{1}{k}\mu^\star(S_h) \le \frac{w_k(S)}{k}, \tag{14}$$

where we have used, in particular, that $\bigcup_{h=0}^{k-1}(k\cdot \mathbf{H}+h)=\mathbf{H}\setminus \mathcal{V}$ for some finite set $\mathcal{V}\subseteq \mathbf{H}$ (to wit, $\mathcal{V}=[\![1,k-1]\!]$ if $\mathbf{H}=\mathbf{N}^+$, and $\mathcal{V}=\varnothing$ otherwise), in such a way that

$$S\setminus \mathcal{V}=(S\cap \mathbf{H})\setminus \mathcal{V}=\bigcup_{h=0}^{k-1}(S\cap (k\cdot \mathbf{H}+h)).$$

Thus, we obtain that

$$\mu_\star(S^c)=1-\mu^\star(S)\geq 1-\frac{w_k(S)}{k}, \tag{15}$$

and it follows by taking $S=X$ in (14) and $S=X^c$ in (15) that

$$1-\frac{w_k(X^c)}{k}\leq \mu_\star(X) \quad \text{and} \quad \mu^\star(X)\leq \frac{w_k(X)}{k}, \tag{16}$$

which, together with Proposition 2(vi), implies (13). ■

Theorem 3. *Let $\mathfrak{b}^\star$ be the upper Buck density on $\mathbf{H}$. We have the following:*

(i) *$\mathfrak{b}^\star$ is an extremal element and the maximum of both $\mathcal{M}^\star(\mathscr{F})$ and $\mathcal{M}^\star(\mathscr{F}_2)$.*
(ii) *If $\mathfrak{b}^\star(X)=0$ for some $X\subseteq \mathbf{H}$, then $\mu^\star(Y)=0$ for every $\mu^\star\in \mathcal{M}^\star(\mathscr{F}_2)$ and $Y\subseteq X$.*

Proof. (i) Clearly $\mathcal{M}^\star(\mathscr{F})\subseteq \mathcal{M}^\star(\mathscr{F}_2)$, and by Example 5 we have that $\mathfrak{b}^\star\in \mathcal{M}^\star(\mathscr{F})$. Thus, we just need to show that $\mathfrak{b}^\star$ is an extremal element and the maximum of $\mathcal{M}^\star(\mathscr{F}_2)$.

To this end, fix a set $X\subseteq \mathbf{H}$. It follows from [31, Theorem 1], which carries over verbatim to the slightly more general version of the upper Buck density considered in the present paper (the main difference lying in the fact that we have $\mathbf{H}$, as the domain of $\mathfrak{b}^\star$, in place of $\mathbf{N}^+$), that there exists an increasing sequence $(k_i)_{i\geq 1}$ of positive integers such that

$$\mathfrak{b}^\star(X)=\lim_{i\to\infty}\frac{r_{k_i}(X)}{k_i},$$

where for $k\geq 1$ we write $r_k(X)$ for the number of residues $h\in [\![0,k-1]\!]$ with the property that $X\cap (k\cdot \mathbf{H}+h)\neq \varnothing$. So Propositions 11 and 6 yield that, for every $\mu^\star\in \mathcal{M}^\star(\mathscr{F}_2)$,

$$\mu^\star(X)\leq \liminf_{k\to\infty}\frac{r_k(X)}{k}\leq \lim_{i\to\infty}\frac{r_{k_i}(X)}{k_i}=\mathfrak{b}^\star(X),$$

which confirms, by the arbitrariness of X, that $\mathfrak{b}^\star$ is the maximum element of $\mathcal{M}^\star(\mathscr{F}_2)$.

As for the rest, suppose to a contradiction that there exist $\mu_1,\mu_2\in \mathcal{M}^\star(\mathscr{F}_2)$ and $\alpha\in]0,1[$ such that $\mu_1\neq\mu_2$ and $\mathfrak{b}^\star=\alpha\mu_1+(1-\alpha)\mu_2$, so that $\mathfrak{b}^\star$ is not an extremal point of $\mathcal{M}^\star(\mathscr{F}_2)$.

Then, by the first part of this proposition, there is no loss of generality in assuming, as we do, that $\mu_1\prec \mathfrak{b}^\star$ and $\mu_2\preceq \mathfrak{b}^\star$, namely, there can be found a set $X\subseteq \mathbf{H}$ such that $\mu_1(X)<\mathfrak{b}^\star(X)$ and $\mu_2(X)\leq \mathfrak{b}^\star(X)$. This is however impossible, because it would imply, together with the fact that $0<\alpha<1$, that $\mathfrak{b}^\star(X)=\alpha\mu_1(X)+(1-\alpha)\mu_2(X)<\mathfrak{b}^\star(X)$.

(ii) Given $Y\subseteq X\subseteq \mathbf{H}$ and $\mu^\star\in \mathcal{M}^\star(\mathscr{F}_2)$, it follows from Remark 1, point (i) above and the monotonicity of $\mathfrak{b}^\star$ (Proposition 4) that $0\leq \mu^\star(Y)\leq \mathfrak{b}^\star(Y)\leq \mathfrak{b}^\star(X)$, which is enough. ■

In fact, Theorem 3 can be dualized by proving that there exists $\mu_\star\in \mathcal{M}_\star(\mathscr{F})$ that is both a minimum and an extremal element of $\mathcal{M}_\star(\mathscr{F}_2)$, where "minimum" means that $\mu_\star\preceq\theta_\star$ for all $\theta_\star\in \mathcal{M}_\star(\mathscr{F}_2)$. We start with a lemma (whose simple proof we omit).

Lemma 4. *Let $(\lambda_\star,\lambda^\star)$ and $(\mu_\star,\mu^\star)$ be conjugate pairs. Then $\lambda^\star\preceq\mu^\star$ if and only if $\mu_\star\preceq\lambda_\star$.*

Corollary 4. *Let $\mathfrak{b}_\star$ be the lower Buck density on* $\mathbf{H}$. *Then $\mathfrak{b}_\star$ is a minimum and an extremal element of both $\mathcal{M}_\star(\mathscr{F})$ and $\mathcal{M}_\star(\mathscr{F}_2)$.*

Proof. Since $\mathcal{M}_\star(\mathscr{F}) \subseteq \mathcal{M}_\star(\mathscr{F}_2)$ and $\mathfrak{b}_\star \in \mathcal{M}_\star(\mathscr{F})$ by Example 5, it is enough to prove that $\mathfrak{b}_\star$ is a minimum and an extremal element of $\mathcal{M}_\star(\mathscr{F}_2)$. But the fact that $\mathfrak{b}_\star$ is a minimum of $\mathcal{M}_\star(\mathscr{F}_2)$ is immediate by Theorem 3 and Lemma 4, and the rest follows by Theorem 3(i). ∎

The next result is a generalization of [31, Theorem 2] and follows at once from Theorem 3(i) and Corollary 4 (we omit further details).

Corollary 5. *If $\mu : \mathcal{P}(\mathbf{H}) \nrightarrow \mathbf{R}$ is a quasi-density on* $\mathbf{H}$, *then* $\mathrm{dom}(\mathfrak{b}) \subseteq \mathrm{dom}(\mu)$ *and* $\mu(X) = \mathfrak{b}(X)$ *for every* $X \in \mathrm{dom}(\mathfrak{b})$, *where $\mathfrak{b}$ is the Buck density on* $\mathbf{H}$.

The question of the existence of minimal elements of $\mathcal{M}^\star(\mathscr{F})$ and $\mathcal{M}^\star(\mathscr{F}_2)$ is subtler, where we say that a function $\mu^\star \in \mathcal{M}^\star(\mathscr{G})$ is minimal if there does not exist any $\theta^\star \in \mathcal{M}^\star(\mathscr{G})$ for which $\theta^\star \prec \mu^\star$. The answer is negative in any model of ZF that admits two or more additive upper densities, and hence in ZFC (see Remark 3). In fact, let $\theta^\star$ be an additive upper density on $\mathbf{H}$, and suppose to a contradiction that $\theta^\star$ is not a minimal element of $\mathcal{M}^\star(\mathscr{F}_2)$, i.e., there exists $\mu^\star \in \mathcal{M}^\star(\mathscr{F}_2)$ such that $\mu^\star \prec \theta^\star$. Then we have from Proposition 2(vi) and Lemma 4 that $\theta_\star \preceq \mu_\star \prec \theta^\star$, where $\theta_\star$ is the lower dual of $\theta^\star$. This is however impossible, because $\theta^\star = \theta_\star$.

Continuing with the notation as above, we now show that $\mathcal{M}^\star(\mathscr{G})$ and $\mathcal{M}_\star(\mathscr{G})$ have at least another notable structural property related to the order $\preceq$. But first we need some terminology.

Specifically, we let a complete upper semilattice be a pair $(L, \le_L)$ consisting of a set L and a (partial) order $\le_L$ on L such that, for every *nonempty* subset S of L, the set

$$\Lambda(S) := \{y \in L : x \le_L y \text{ for all } x \in S\},$$

has a least element, viz. there exists $y_0 \in \Lambda(S)$ such that $y_0 \le_L y$ for every $y \in \Lambda(S)$; cf., e.g., [12, §§ 1.10 and 3.14], where the condition $S \ne \varnothing$ is not assumed. On the other hand, we say that $(L, \le_L)$ is a complete lower semilattice if and only if $(L, \ge_L)$ is a complete upper semilattice, where $\ge_L$ is the partial order on L defined by taking $x \ge_L y$ if and only if $y \le_L x$.

Proposition 12. *$(\mathcal{M}^\star(\mathscr{G}), \preceq)$ is a complete upper semilattice.*

Proof. Pick a nonempty subset S of $\mathcal{M}^\star(\mathscr{G})$, and let $\theta^\star$ denote the function $\mathcal{P}(\mathbf{H}) \to \mathbf{R} : X \mapsto \sup_{\mu^\star \in S} \mu^\star(X)$, which is well defined because $S \ne \varnothing$ and $\mathrm{Im}(\mu^\star) \subseteq [0,1]$ for every $\mu^\star \in S$.

It is clear that $\mathrm{Im}(\theta^\star) \subseteq [0,1]$ and $\mu^\star \preceq \theta^\star$ for every $\mu^\star \in S$. In addition, if $\mu^\star(\mathbf{H}) = 1$ for every $\mu^\star \in \mathcal{M}^\star(\mathscr{G})$ then $\theta^\star(\mathbf{H}) = 1$ too, and if every $\mu^\star \in \mathcal{M}(\mathscr{G})$ is subadditive then

$$\theta^\star(X \cup Y) \le \sup_{\mu^\star \in S} (\mu^\star(X) + \mu^\star(Y)) \le \sup_{\mu^\star \in S} \mu^\star(X) + \sup_{\mu^\star \in S} \mu^\star(Y) = \theta^\star(X) + \theta^\star(Y),$$

for all $X, Y \subseteq \mathbf{H}$ (i.e., also $\mu^\star$ is subadditive). Similarly, $\theta^\star$ is monotone, (-1)-homogeneous, or translational invariant, respectively, if so is every $\mu^\star \in \mathcal{M}^\star(\mathscr{G})$ (we omit details). ∎

Incidentally, Proposition 12 implies that both $\mathcal{M}^\star(\mathscr{F})$ and $\mathcal{M}^\star(\mathscr{F}_2)$ have a maximum element (not necessarily the same), but does not identify it more precisely, in contrast to Theorem 3(i). Similar considerations apply also to the following result, when compared with Corollary 4 (again, we leave the proof as an exercise for the reader).

Corollary 6. *$(\mathcal{M}_\star(\mathscr{G}), \preceq)$ is a complete lower semilattice.*

Lastly, let S be a set and $(A, \le_A)$ a directed preordered set. To wit, A is a set and $\le_A$ is a reflexive and transitive binary relation on A such that for every nonempty finite $B \subseteq A$ there is $\alpha \in A$ with $\beta \le_A \alpha$ for all $\beta \in B$. A net $(f_\alpha)_{\alpha \in A}$ of functions $S \to \mathbf{R}$ is any function $\eta : A \to \mathbf{R}^S$ (the set of all functions $S \to \mathbf{R}$). We say that the net $(f_\alpha)_{\alpha\in A}$ is pointwise convergent if there exists a function $f : S \to \mathbf{R}$, which we call a pointwise limit of $(f_\alpha)_{\alpha\in A}$, such that for every $x \in S$ the real net $(f_\alpha(x))_{\alpha\in A}$ converges to $f(x)$ with respect to the usual topology on $\mathbf{R}$.

With these definitions and the above notation in hand, we have the following proposition, whose proof is left as an exercise for the reader.

Proposition 13. *Let $(\mu_\alpha)_{\alpha\in A}$ and $(\lambda_\alpha)_{\alpha\in A}$ be, respectively, pointwise convergent nets with values in $\mathcal{M}^\star(\mathscr{G})$ and $\mathcal{M}_\star(\mathscr{G})$, and denote by μ a pointwise limit of $(\mu_\alpha)_{\alpha\in A}$ and by λ a pointwise limit of $(\lambda_\alpha)_{\alpha\in A}$. Then μ and λ are unique and belong, respectively, to $\mathcal{M}^\star(\mathscr{G})$ and $\mathcal{M}_\star(\mathscr{G})$.*

We conclude the section by adding one more distinguished item to the list of upper densities.

EXAMPLE 8. Let $\mathfrak{p}^\star$ be the upper Pólya density on $\mathbf{H}$, viz. the function

$$\mathcal{P}(\mathbf{H}) \to \mathbf{R} : X \mapsto \lim_{s\to 1^-} \limsup_{n\to\infty} \frac{|X \cap [1,n]| - |X \cap [1,ns]|}{(1-s)n}.$$

It is not difficult to check that the dual of $\mathfrak{p}^\star$ is the function

$$\mathfrak{p}_\star : \mathcal{P}(\mathbf{H}) \to \mathbf{R} : X \mapsto \lim_{s\to 1^-} \liminf_{n\to\infty} \frac{|X \cap [1,n]| - |X \cap [1,ns]|}{(1-s)n},$$

which we refer to as the lower Pólya density on $\mathbf{H}$. Among other things, $\mathfrak{p}^\star$ has found a number of remarkable applications in analysis, see e.g. [35] and [23], but what is perhaps more interesting in the frame of the present work is that $\mathfrak{p}^\star$ is an upper density in the sense of our definitions: This follows from Proposition 13 and the fact that $\mathfrak{p}^\star$ is the pointwise limit of the real net of the α-densities on $\mathbf{H}$, see [22, Theorem 4.3].

7. CLOSING REMARKS AND OPEN QUESTIONS

Below, we draw a list of questions we have not been able to answer, some of them being broad generalizations of popular questions from the literature on densities.

Question 3. Let $\mu^\star$ be an upper quasi-density on $\mathbf{H}$ and $\mu_\star$ its conjugate, and for every $X \subseteq \mathbf{H}$ denote by $\mathfrak{D}_X(\mu^\star)$ the set of all pairs $(a_1, a_2) \in \mathbf{R}^2$ such that $a_1 = \mu_\star(Y)$ and $a_2 = \mu^\star(Y)$ for some $Y \subseteq X$. Is $\mathfrak{D}_X(\mu^\star)$ a convex or closed subset of $\mathbf{R}^2$ for every $X \subseteq \mathbf{H}$?

Notice that if $\mu^\star$ is an upper density then $\mathfrak{D}_X(\mu^\star)$ is contained, by Proposition 2(vi), in the trapezium $\{(a_1, a_2) \in [0,1]^2 : 0 \le a_1 \le \mu_\star(X)$ and $a_1 \le a_2 \le \mu^\star(X)\}$, but this is no longer the case when $\mu^\star$ does not satisfy (F2), as it follows from Theorem 1.

In fact, the answer to Question 3 is positive when $\mu^\star$ is, for some real exponent $\alpha \ge -1$, the classical upper α-density on $\mathbf{N}^+$, as essentially proved in [14, 15]. More in general, the same is true for certain upper weighted densities, as we get from [17, Theorem 2].

On a related note, we ask the following question, which has a positive answer for the Buck density and the Banach density, see [32, Theorem 2.1] and [10, Theorem 4.2], respectively.

Question 4. Let μ be a quasi-density on $\mathbf{H}$. Given $X \in \operatorname{dom}(\mu)$ and $a \in [0, \mu(X)]$, does there exist $Y \in \operatorname{dom}(\mu)$ such that $Y \subseteq X$ and $\mu(Y) = a$?

In fact, the existence of upper quasi-densities that are not upper densities (Theorem 1) raises a number of questions. In particular, Theorem 3(ii) suggests the following:

Question 5. Let $\mu^\star$ be an upper quasi-density on $\mathbf{H}$. If $X \subseteq \mathbf{H}$ and $\mu^\star(X) = 0$, can there exist a set $Y \subseteq X$ for which $\mu^\star(Y) \neq 0$? And in the same spirit, is it possible that $\mu^\star(X) \neq \mu^\star(Y)$ for some $X, Y \subseteq \mathbf{H}$ such that $|X \triangle Y| < \infty$?

The case of upper densities is covered by the following supplement to Proposition 2(v):

Proposition 14. *Let $\mu^\star$ be an upper density on $\mathbf{H}$ and $\mu_\star$ its conjugate, and pick $X, Y \subseteq \mathbf{H}$. If $|X \triangle Y| < \infty$, then $\mu^\star(X) = \mu^\star(Y)$ and $\mu_\star(X) = \mu_\star(Y)$.*

Proof. Assume that $|X \triangle Y| < \infty$. By Proposition 6, we have $\mu^\star(X \triangle Y) = 0$, and since $X \triangle Y = X^c \triangle Y^c$, this implies, along with Proposition 1(i), that $\mu^\star(X) = \mu^\star(Y)$ and $\mu^\star(X^c) = \mu^\star(Y^c)$. Therefore, we find that $\mu_\star(X) = 1 - \mu^\star(X^c) = 1 - \mu^\star(Y^c) = \mu_\star(Y)$. ∎

A special case of Question 5, which would simplify some of the proofs of this paper and show that the case $\mathbf{H} = \mathbf{N}^+$ can be reduced, to some degree, to the case $\mathbf{H} = \mathbf{N}$, is as follows:

Question 6. Can every upper quasi-density on $\mathbf{N}^+$ be *uniquely* extended to one on $\mathbf{N}$?

We obtain from Propositions 1(iv) and 6 that if we replace upper quasi-densities with upper densities in Questions 5 and 6, then the answer to the former is negative and the answer to the latter is positive. The same argument leads to the next proposition (we omit further details), which shows, in particular, that lower densities are translational invariant.

Proposition 15. *Let $(\mu_\star, \mu^\star)$ be a conjugate pair on $\mathbf{H}$ such that $\mu^\star$ satisfies (F2), (F3) and (F5). Then $\mu_\star$ is translational invariant.*

On the other hand, we have not been able to answer the following:

Question 7. Is it true that every lower density is (-1)-homogeneous? And is there a lower quasi-density that is not translational invariant?

On a different note, we may ask if a set $X \subseteq \mathbf{H}$ such that $\mu^\star(X) > 0$ for some upper density $\mu^\star$, has to contain a finite arithmetic progression of arbitrary length. The answer is negative: It follows from [32, Theorem 3.2] that the upper Buck density of the set $X := \{n + n! : n \in \mathbf{N}\}$ is 1, yet X does not contain any finite arithmetic progression of length 3.

Loosely speaking, this means that neither Szemerédi's theorem [45] nor Roth's theorem [36, 37] are characteristic of the theory of upper densities, as long as the notion of upper density is interpreted in the lines of the present paper, and it could be interesting to answer the following:

Question 8. Does there exist a reasonable set of axioms, alternative to or sharper than (F1)-(F6), for which an "abstract version" of Roth's or Szemerédi's theorem can be proved?

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REFERENCES

[1] R. Alexander, *Density and multiplicative structure of sets of integers*, Acta Arith. **12** (1967), 321–332.

[2] A. Blass, R. Frankiewicz, G. Plebanek, and C. Ryll-Nardzewski, *A note on extensions of asymptotic density*, Proc. Amer. Math. Soc. **129** (2001), No. 11, 3313–3320.

[3] V. Bergelson and T. Downarowicz, *Large sets of integers and hierarchy of mixing properties of measure preserving systems*, Colloq. Math. **110** (2008), No. 1, 117–150.

[4] R. C. Buck, *The Measure Theoretic Approach to Density*, Amer. J. Math. **68** (1946), No. 4, 560–580.

[5] ———, *Generalized asymptotic density*, Amer. J. Math. **75** (1953), No. 2, 335–346.

[6] M. Di Nasso, *Fine asymptotic densities for sets of natural numbers*, Proc. Amer. Math. Soc. **138** (2010), No. 8, 2657–2665.

[7] E. K. van Douwen, *Finitely additive measures on* $\mathbb{N}$, Topology Appl. **47** (1992), No. 3, 223–268.

[8] A. R. Freedman and J. J. Sember, *Densities and summability*, Pacific J. Math. **95** (1981), No. 2, 293–305.

[9] A. Fuchs and R. Giuliano Antonini, *Théorie générale des densités*, Rend. Accad. Naz. Sci. XL Mem. Mat. **14** (1990), No. 1, 253–294.

[10] Z. Gáliková, B. László, and T. Šalát, *Remarks on uniform density of sets of integers*, Acta Acad. Paed. Agriensis, Sectio Mathematicae **29** (2002), 3–13.

[11] R. Giuliano-Antonini and G. Grekos, *Comparison between lower and upper* α*-densities and lower and upper* α*-analytic densities*, Unif. Distrib. Theory **3** (2008), No. 2, 21–35.

[12] G. Grätzer, *Lattice Theory: Foundation*, Birkhäuser: Basel, 2011.

[13] B. Green and T. Tao, *The primes contain arbitrarily long arithmetic progressions*, Ann. of Math. **167** (2008) No. 2, 481–547.

[14] G. Grekos, *Sur la répartition des densités des sous-suites d'une suite d'entiers*, PhD thesis, Université Pierre et Marie Curie, 1976 (in French).

[15] ———, *Répartition des densités des sous-suites d'une suite d'entiers*, J. Number Theory **10** (1978), No. 2, 177–191 (in French).

[16] ———, *On various definitions of density (survey)*, Tatra Mt. Math. Publ. **31** (2005), 17–27.

[17] G. Grekos, L. Mišík, and J. T. Tóth, *Density sets of sets of positive integers*, J. Number Theory **130** (2010), No. 6, 1399–1407.

[18] G. Grekos, V. Toma, and J. Tomanová, *A note on uniform or Banach density*, Ann. Math. Blaise Pascal **17** (2010), No. 1, 153–163.

[19] H. Halberstam and K. F. Roth, *Sequences*, Springer-Verlag: Berlin, 1983 (revised edition).

[20] R. Jin, "Density Versions of Plünnecke Inequality: Epsilon-Delta Approach", 99–113 in: M. B. Nathanson (ed.), *Combinatorial and Additive Number Theory - CANT 2011 and 2012*, Springer Proc. Math. Stat. **101**, Springer: New York, 2014.

[21] P. Leonetti and S. Tringali, *Upper and lower densities have the strong Darboux property*, J. Number Theory **174** (2017), 445–455.

[22] P. Letavay, L. Mišík, and M. Sleziak, *Extreme points of the set of density measures*, J. Math. Anal. Appl. **423** (2015), No. 2, 1150–1165.

[23] N. Levinson, *Gap and density theorems*, Amer. Math. Soc. Colloq. Publ. **26**, New York, 1940.

[24] A. A. Liapounoff, *Sur les fonctions-vecteurs complètement additives*, Izv. Akad. Nauk SSSR Ser. Mat. **4** (1940), No. 6, 465–478.

[25] D. Maharam, *Finitely additive measures on the integers*, Sankhyā Ser. A **38** (1976), No. 1, 44–59.

[26] H. B. Mann, *A proof of the fundamental theorem on the density of sums of sets of positive integers*, Ann. of Math. (2) **43** (1942), No. 3, 523–527.

[27] A. H. Mekler, *Finitely additive measures on* $\mathbf{N}$ *and the additive property*, Proc. Amer. Math. Soc. **92** (1984), No. 3, 439–444.

[28] M. B. Nathanson, *Elementary Methods in Number Theory*, Grad. Texts in Math. **195**, Springer: New York, 2000.

[29] I. Niven, *The asymptotic density of sequences*, Bull. Amer. Math. Soc. **57** (1951), No. 6, 420–434.

[30] I. Niven, H. S. Zuckerman, and H. L. Montgomery, *An introduction to the theory of numbers*, Wiley, 1991 (5th edition).

[31] M. Paštéka, *Some properties of Buck's measure density*, Math. Slovaca **42** (1992), No. 1, 15–32.

[32] M. Paštéka and T. Šalát, *Buck's measure density and sets of positive integers containing arithmetic progression*, Math. Slovaca **41** (1991), No. 3, 283–293.

[33] M. Paštéka, T. Šalát, and T. Visnyai, *Remarks on Buck's measure density and a generalization of asymptotic density*, Tatra Mt. Math. Publ. **31** (2005), 87–101.

[34] D. Pincus, "The strength of the Hahn-Banach theorem", 203–248 in: A. E. Hurd and P. Loeb (eds.), *Victoria Symposium on Nonstandard Analysis*, Lecture Notes in Math. **369**, Springer: Berlin, 1974.

[35] G. Pólya, *Untersuchungen über Lücken und Singularitäten von Potenzreihen*, Math. Zeit., **29** (1929), 549–640.

[36] K. F. Roth, *Sur quelques ensembles d'entiers*, C. R. Acad. Sci. Paris **234** (1952), 388–390.

[37] ———, *On certain sets of integers*, J. London Math. Soc. **28** (1953), 104–109.

[38] I. Z. Ruzsa, "Sumsets and Structure", 87–210 in: A. Geroldinger and I. Z. Ruzsa, *Combinatorial Number Theory and Additive Group Theory*, Advanced Courses in Mathematics - CRM Barcelona, Birkhäuser: Basel, 2009.

[39] T. Šalát and R. Tijdeman, *Asymptotic densities of sets of positive integers*, Math. Slovaca, **33** (1983), No. 2, 199–207.

[40] E. Schechter, *Handbook of Analysis and its Foundations*, Academic Press, 1996.

[41] ———, *Elementary Theory of Numbers*, North-Holland Mathematical Library **31**, North-Holland: Amsterdam, 1988 (2nd edition).

[42] M. Sleziak and M. Ziman, *Lévy group and density measures*, J. Number Theory **128** (2008), No. 12, 3005–3012.

[43] ———, *Range of density measures*, Acta Math. Univ. Ostrav. **17** (2009), No. 1, 33–50.

[44] A. Soifer, *The Mathematical Coloring Book*, Springer-Verlag: New York, 2009.

[45] E. Szemerédi, *On sets of integers containing no k elements in arithmetic progression*, Acta Arith. **27** (1975), 199–245.

[46] G. Tenenbaum, *Introduction to Analytic and Probabilistic Number Theory*, Cambridge Stud. Adv. Math. **46**, Cambridge Univ. Press: Cambridge, 1995.

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